Imperfect Bifurcation in Structures and Materials — Engineering Use of Group-Theoretic Bifurcation Theory

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Preface

Many physical systems lose or gain stability and pattern through bifurcation behavior. Extensive research of this behavior is carried out in many fields of science and engineering. The study of dynamic bifurcation behavior, for example, has made clear the mechanism of dynamic instability and chaos. The group-theoretic bifurcation theory is an established means to deal with the formation and selection of patterns in association with symmetry-breaking bifurcation. Since all physical systems are "imperfect," in that they inevitably involve some initial imperfections, the study of imperfect bifurcation (bifurcation of imperfect systems) has drawn a keen mathematical interest to yield a series of important results, such as the universal unfolding.

In structural mechanics, bifurcation behavior has been studied to model the buckling and failure of structural systems. The sharp reduction of the strength of structural systems by initial imperfections is formulated as imperfection sensitivity laws. A series of statistical studies has been conducted to make clear the dependence of the strength of structures on the statistical variation of initial imperfections. A difficulty in these studies arises from the presence of a large number of initial imperfections. At this state, most of these studies are carried out based on the Monte Carlo simulation for a number of initial imperfections, or, on an imperfection sensitivity law against a single initial imperfection.

In geomechanics, predominant role of bifurcation behavior in strengths and deformation patterns of the geomaterials, sand and soil, has come to be acknowledged. Yet the experimental behavior of geomaterials is quite obscured by the presence of initial imperfections; moreover, observed curves of force versus displacement can be qualitatively different from the bifurcation diagrams predicted by mathematics. Although many defects in geomaterials are well-known to form some geometrical patterns, the underlying mechanism of these patterns still remains open.

To sum up, notwithstanding extensive studies of bifurcation behavior in many fields of research, there seems to be a gap between the mathematical theory and engineering practice. In an attempt to fill this gap we offer, in this book, the modern view on static imperfect bifurcation behavior.

Major objectives of this book are:

- to develop theories on the strength variation of (structural) systems due to initial imperfections;
- to develop a systematic technique to deal with bifurcation diagrams to be observed in experiments as opposed to conceptual and schematic diagrams in mathematics; and
- to develop a method to reveal the mechanism of patternized defects of uniform materials.

These objectives are achieved on the basis of a series of works by the authors that serves as an extension of the basic tools:

- the asymptotic bifurcation theory;
- the statistical approach to random initial imperfections; and
- the group-theoretic bifurcation theory.

This book consequently offers a wider and deeper insight into imperfect bifurcation behavior in engineering problems. Our approach to imperfections is pragmatic, rather than mathematically rigorous, and is intended to be an introduction for students in engineering by minimizing the mathematical formalism. This book will be of assistance to mathematicians as well, showing way how bifurcation theory is to be made useful for actual problems. A proper modeling of symmetries of systems, for example, will lead to a proper understanding of their bifurcation behavior with the aid of group-theoretic bifurcation theory. This book offers a number of systematic methods, based on up-to-date mathematics, to untangle the mechanism of real physical and structural problems undergoing bifurcation, such as soil, sand, kaolin, concrete, and regular-polygonal domes. The present approach to the elastic bifurcation is successfully applied to the experimental behaviors of materials; in particular, the symmetry-breaking bifurcation behaviors of uniform materials is introduced as an essential source of the emergence of patterns on the surface of materials. The horizon of static bifurcation has thus been extended.

Bifurcation is associated with an instability induced by a singular Jacobian matrix of a system—the linearized eigenvalue problem. A critical (singular) point is the one at which one or more eigenvalues of this matrix vanish, and it is at such a point (with some additional conditions) where bifurcation actually takes place. According to whether the number of zero eigenvalue(s) is equal to or greater than one, the critical point is classified into two types:

- a simple critical point; and
- a multiple critical point.

The bifurcation behavior of the multiple critical point is far more complex than that of the simple critical point in that more paths can potentially branch.

This book is divided into three parts. In Part I we aim at the fundamental understanding of the concepts and theories of initial imperfections with reference to simple structural models, focusing on simple critical points. In Part II we extend them to systems with geometrical symmetries, for which multiple criticality appears generically, with reference to more realistic examples. In Part III we tackle the bifurcation behavior of physical and structural systems with various kinds of symmetries and, in turn, address the issue of modeling symmetries of these systems. The contents of this book are outlined below.

Chapter 1: Introduction to Bifurcation Behavior. This first chapter offers an introduction to bifurcation behavior. A few examples of bifurcation are presented to clarify the mechanism of bifurcation and the influence of initial imperfections. Furthermore, an overview of the book is presented to highlight important results of the book.

Part I: Imperfect Behavior at Simple Critical Points. This first part is devoted to the study of imperfect behavior in the vicinity of a simple critical point. With the help of the simplicity due to simple criticality, the fundamental characteristics of imperfect behavior are investigated in an asymptotic sense. Here the word "asymptotic" means that the results are valid in a sufficiently close neighborhood of the critical point under consideration for a sufficiently small value of initial imperfection(s). Various important aspects of imperfect behavior, such as the bifurcation equation, imperfection sensitivity, critical (worst) imperfection, probabilistic variation, and observability are introduced in Chapters 2 through 6, respectively. Emphasis is placed on the case of a large number of initial imperfection parameters, while it is often the case in the literature to deal with one or two parameters.

Part II: Imperfect Bifurcation of Symmetric Systems. In this second part, we extend the results of Part I to multiple critical points of symmetric systems. In order to avoid sophisticated mathematical concepts, we focus on the simplest groups, the dihedral and cyclic groups that label in-plane symmetries. Nonetheless, the basic strategy presented is general enough and is extendible to other groups. In Chapter 7 group-theoretic bifurcation theory is briefly introduced as a mathematical tool to deal with group-theoretic degeneracy. In Chapter 8 we present the theory of perfect and imperfect bifurcation behaviors in the vicinity of a critical point of a system with dihedral or cyclic group symmetry. This theory is applied to spherical truss dome structures. The critical imperfection and probabilistic variation of imperfections are studied in Chapters 9 and 10, respectively. In Chapter 11 perfect and imperfect behaviors of more realistic systems, such as truss dome structures and soil specimens, are investigated by means of a synthetic application of the procedures presented in Part II.

Part III: Modeling of Bifurcation Phenomena. In this third part, we study the bifurcation behaviors of various kinds of physical and structural systems by modeling their symmetries appropriately. In Chapter 12 the recursive change of the shapes of cylindrical sand specimens undergoing bifurcation is investigated. In Chapter 13 the mechanism of echelon-mode formation on sand, kaolin and steel specimens is revealed by investigating the bifurcation of an $O(2)\times O(2)$-equivariant system. In Chapter 14 the recursive bifurcation of rectangular parallelepiped steel specimens is studied. In Chapter 15 miscellaneous aspects of the bifurcation behaviors of materials that have been left over are presented to show the diversity of bifurcation behaviors.

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