9-Intersection Calculi for Spatial Reasoning on the Topological Relations between Multi-Domain Objects

Yohei Kurata

SFB/TR8 Spatial Cognition, Universität Bremen
Postfach 330 440, 28334 Bremen, Germany
ykurata@informatik.uni-bremen.de

Abstract

Most spatial calculi target spatial relations between single-type objects, whereas there are also a number of spatial models that distinguish spatial relations between objects in different domains. How to equip such cross-domain spatial models with reasoning capability is left as a research question. As a first step, this paper develops a series of qualitative spatial calculi based on the 9-intersection. The 9-intersection distinguishes topological relations between various objects (points, lines, regions, bodies, etc.). We formulate two sorts of calculi: *homogeneous 9-intersection calculi* target the topological relations between single-type objects, while *heterogeneous 9-intersection calculi* can deal with multiple sets of topological relations between various combinations of objects. As the foundation of these calculi, composition tables and lists of converse relations are developed for various sets of topological relations in $\mathbb{R}^1$ and $\mathbb{R}^2$. For heterogeneous 9-intersection calculi, the sets of base relations, composition tables, and list of converse relations are integrated, such that the algebraic framework of ordinary single-domain spatial calculi can be reused. Finally, the use of the new calculi is demonstrated.

Keywords: qualitative spatial calculi, topological relations, cross-domain spatial models, 9-intersection, composition tables

1. Introduction

Qualitative spatial calculi provide reasoning capability for the models of spatial relations. With these calculi, for instance, incompletely-observed spatial arrangements of objects can be disambiguated with regard to a specific set of spatial relations. Interestingly, most of existing qualitative spatial calculi target spatial relations between single-sort objects. For instance, Allen's interval algebra [1], Region Connection Calculus [2], Cardinal Direction Calculus [3], and Double Cross Calculus [4] feature a set of relations between two intervals, two regions, two points, and three points, respectively. Such *single-domain spatial calculi* fit nicely into an algebraic framework (*relation algebra* or its family), since their operations are closed under the single set of spatial relations. On the other hand, spatial database communities have developed a number of spatial models that distinguish spatial relations between objects in different domains. For instance, the 9-intersection [5] can distinguish the topological relations between a line and a region, in addition to the relations between two regions or those between two lines. In [6], the target of cardinal direction relations is extended to arbitrary combination of objects (points, lines, and regions). In [7], spatial arrangements of a path and a landmark are modeled as relations between a directed line and a region using a Double-Cross-like frame of spatial reference. How to equip such cross-domain spatial models with reasoning capability is left as a research question. For instance, imagine a space where point-like objects, line-like objects, and region-like objects coexist. Given partial knowledge about their arrangement, can we disambiguate it? If this is possible by computation, the applicability of qualitative spatial calculi will be expanded considerably.

As a first step, this paper develops a series of qualitative spatial calculi based on the 9-intersection [5]. These *9-intersection calculi* consist of *homogeneous 9-intersection calculi*, which target a set of topological relations between single-type objects (e.g., line-line relations), and *heterogeneous 9-intersection calculi*, which can deal with multiple sets of topological relations between various combinations of objects (e.g., mixture of line-line, line-region, and region-region relations). We show that the algebraic framework of ordinary single-domain spatial calculi can be reused for the heterogeneous 9-intersection calculi and, consequently, we can use existing reasoning tools of spatial calculi, such as SparQ [8] and GQR [9], to conduct reasoning in the heterogeneous 9-intersection calculi. We expect that a similar approach achieves reasoning capability in other cross-domain spatial models as well.

A secondary but important challenge of this paper is to develop composition tables for various combinations of topological relations. The composition table of two topological region-region relations in $\mathbb{R}^2$ is reported in [10], but the tables for other combinations are not fully developed yet. We therefore develop these composition tables i systematically with a small number of composition rules.

The remainder of this paper is organized as follow: Sections 2 and 3 summarize major concepts of qualitative spatial calculi and the 9-intersection, respectively. Section 4 develops the lists of converse relations and composition tables for various topological relations. Section 5 develops the 9-intersection calculi based on these lists and composition tables. Section 6 demonstrates the use of these calculi for qualitative spatial reasoning. Finally, Section 7 concludes with a discussion of future problems.

2. Qualitative Spatial Calculi

Qualitative spatial calculi (and their lower dimensional counterparts, qualitative temporal calculi) have been studied extensively in AI communities [11, 12]. In a broad sense, qualitative spatial calculi are the calculi formed by a set of spatial relations and operations on these relations. Typically, binary spatial calculi are equipped with two operations, *conversion* (*converse*) and *composition*, in addition to set-theoretic operations. By conversion we can derive the relation between A and B from the relation between B and A, while composition enable us to derive possible relations between A and B from the relation between A and C and that between C and B. Ternary spatial calculi also have counterparts of these operations [13]. This paper focuses on binary spatial calculi, since topological relations are binary relations.

Normally each binary spatial calculus targets a jointly exclusive and pairwise disjoint set of spatial relations that may hold between two arbitrary objects in a spatial object domain **D** (points, regions, etc.), including an identity relation. These spatial relations are called *base relations* and as a set they are denoted $\boldsymbol{\mathcal{B}}$.

In order to process incomplete knowledge about spatial relations, the set of all base relations that may hold between a pair of objects is treated as a unit of computation, called (*general*) *relation*. For instance, if the topological relations between two regions A and B are known to be $\textit{disjoint}_{\mathrm{RR}}$ or $\textit{meet}_{\mathrm{RR}}$, the relation between A and B is represented as $\{\textit{disjoint}_{\mathrm{RR}}, \textit{meet}_{\mathrm{RR}}\}$. If nothing is known about the possible spatial relations between A and B, the relation between A and B is represented by the set of all base relations in $\boldsymbol{\mathcal{B}}$, which is called the *universal relation* and denoted $\boldsymbol{U}$.

The set of all relations (essentially $\boldsymbol{\mathcal{B}}$'s power-set $\mathbf{2}^{\boldsymbol{\mathcal{B}}}$) is denoted $\boldsymbol{\mathcal{R}}$. The converse $^{\cup}$ and the composition ; on $\boldsymbol{\mathcal{R}}$ are defined based on those on $\boldsymbol{\mathcal{B}}$ as equations 1-2. The set $\boldsymbol{\mathcal{R}}$, together with its converse and composition operations closed under $\boldsymbol{\mathcal{R}}$, gives rise to an algebra. Normally, a binary spatial calculus forms a *non-associative algebra* (or even its stronger version, a *relation algebra* or a *semi-associative algebra*, depending on its associativity [12]). Actually, from an algebraic point of view, Ligozat and Renz [12] defined a qualitative binary spatial calculus as a tuple of a non-associative algebra and its weak representation.

$$\forall R \in \boldsymbol{\mathcal{R}}\; R^{\cup} = \bigcup_{r \in R} r^{\cup} \tag{1}$$

$$\forall R_1, R_2 \in \boldsymbol{\mathcal{R}}\; R_1; R_2 = \bigcup_{r_i \in R_1, r_j \in R_2} (r_i; r_j) \tag{2}$$

The merit of such an algebraic treatment is that we can computationally disambiguate the relations between many objects by algebraic computation without paying attention to actual geometry of the objects. This problem corresponds to a constraint satisfaction problem (CSP). The CSP's key question is consistency checking, i.e., to identify the presence or absence of the variables that satisfy the given constraints. In spatial calculi, the variables and constraints correspond to spatial objects and their relations, respectively. Through checking algebraic closeness of every scenario, we can detect invalid combinations of spatial relations that cannot hold between the objects (or the absence of such combinations). By filtering them out, we can derive the *candidates* for the possible combinations of spatial relations between the objects (although at this level we cannot guarantee that all of these candidates are geometrically realizable). There are already some effective tools to support such constraint-based reasoning on user-defined spatial/temporal (e.g., SparQ [8] and GQR [9]).

3. The 9-Intersection

The 9-intersection [5] is a model of binary topological relations based on point-set topology [14]. This model has been studied extensively in spatial database communities, primarily because it applies to various combinations of objects systematically. In this model, the relations between two objects are distinguished by certain properties of intersections between their *topological parts* (*interior*, *boundary*, and *exterior*). The interior, boundary, and exterior of a spatial object X, denoted X°, ∂X, and X^{-}, are defined as the union of all open sets contained in X, the difference between X's *closure* (i.e., the intersection of all closed point sets that contain X) and X°, and the complement of X's closure, respectively. The *9-intersection matrix* in equation 3 concisely represents the 3×3 parts' intersections between two objects A and B.

$$\mathrm{M}(A, B) = \begin{pmatrix} A^{\circ} \cap B^{\circ} & A^{\circ} \cap \partial B & A^{\circ} \cap B^{-} \\ \partial A \cap B^{\circ} & \partial A \cap \partial B & \partial A \cap B^{-} \\ A^{-} \cap B^{-} & A^{-} \cap \partial B & A^{-} \cap B^{-} \end{pmatrix} \tag{3}$$

In the most basic approach, topological relations are distinguished by the presence or absence of these 3×3 intersections. Thus, we consider *two-valued 9-intersection matrix* whose element are either empty ($\emptyset$) or non-empty ($\neg\emptyset$). By the patterns of the two-valued 9-intersection matrix, for instance, we can distinguish two point-point relations, three point-line relations, and eight line-line relations in a 1D Euclidian space $\mathbb{R}^1$ (figure 1). Note that by definition a point does not have an interior and the line's boundary refers to the set of its two endpoints.

The set of topological relations distinguished by the patterns of two-valued 9-intersection is denoted $\boldsymbol{\mathcal{T}}_{\mathbf{D}_1\mathbf{D}_2\text{-}\mathcal{S}}$, where $\mathbf{D}_1$ and $\mathbf{D}_2$ are the domains of two objects ($\mathbf{P}$: points, $\mathbf{L}$:simple lines, $\mathbf{R}$:simple regions, and $\mathbf{B}$:simple bodies) and $\mathcal{S}$ is the space. For instance, $\boldsymbol{\mathcal{T}}_{\mathbf{LL}\text{-}\mathbb{R}^1}$ refers to the set of topological line-line relations in $\mathbb{R}^1$ (figure 1d). Table 1 summarizes the numbers of topological relations distinguished by the patterns of the two-valued 9-intersection matrix.

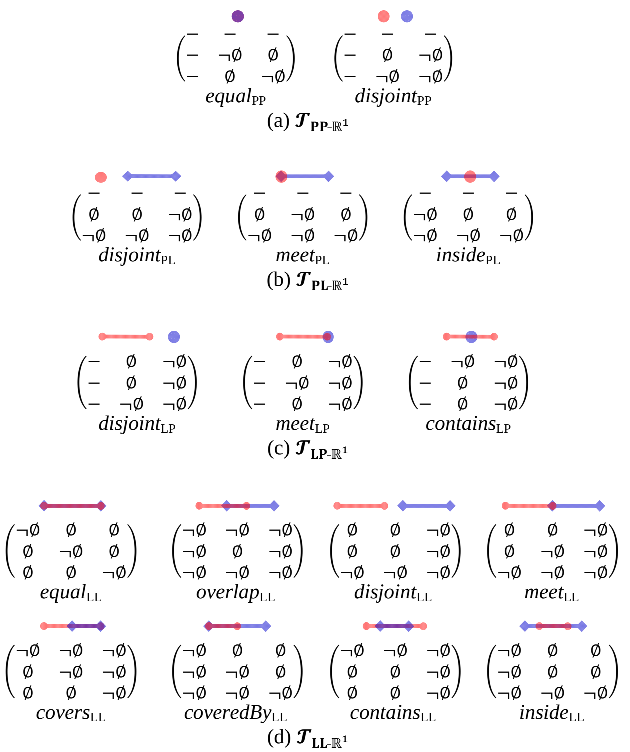


Figure 1. Topological relations between points, lines, and their combinations in $\mathbb{R}^1$ distinguished by the patterns of the two-valued 9-intersection.

Table 1. Numbers of topological relations distinguished by the patterns of the two-valued 9-intersection matrix [15].

	$\mathbb{R}^1$	$\mathbb{R}^2$	$\mathbb{R}^3$	$\mathbb{S}^1$	$\mathbb{S}^2$
Point-Point	2	2	2	2	2
Point-Line / Line-Point	3	3	3	3	3
Point-Region / Region-Point	–	3	3	–	3
Point-Body / Body-Point	–	–	3	–	–
Line-Line	8	33	33	11	33
Line-Region / Region-Line	–	19	31	–	19
Line-Body / Body-Line	–	–	19	–	–
Region-Region	–	8	43	–	11
Region-Body / Body-Region	–	–	19	–	–
Body-Body	–	–	8	–	–

4. Conversion and Composition

Conversion and composition are fundamental operations of qualitative spatial calculi. This section develops these two operations for a variety of topological relations.

The converse of a relation r in $\boldsymbol{\mathcal{T}}_{\mathbf{D}_1\mathbf{D}_2\text{-}\mathcal{S}}$ is a relation in $\boldsymbol{\mathcal{T}}_{\mathbf{D}_2\mathbf{D}_1\text{-}\mathcal{S}}$. For instance, the converse of *contains*$_{\mathrm{RP}}$ in $\boldsymbol{\mathcal{T}}_{\mathbf{RP}\text{-}\mathbb{R}^2}$ (region-point relations) is *inside*$_{\mathrm{PR}}$ in $\boldsymbol{\mathcal{T}}_{\mathbf{PR}\text{-}\mathbb{R}^2}$ (point-region relations). We can derive the converse of a relation r in $\boldsymbol{\mathcal{T}}_{\mathbf{D}_1\mathbf{D}_2\text{-}\mathcal{S}}$ simply by transposing r's 9-intersection matrix and finding the same pattern from the two-valued 9-intersection matrices that represent the relations in $\boldsymbol{\mathcal{T}}_{\mathbf{D}_2\mathbf{D}_1\text{-}\mathcal{S}}$. By repeating this process for every relation in $\boldsymbol{\mathcal{T}}_{\mathbf{D}_1\mathbf{D}_2\text{-}\mathcal{S}}$, we can obtain the *converse list* of $\boldsymbol{\mathcal{T}}_{\mathbf{D}_1\mathbf{D}_2\text{-}\mathcal{S}}$, denoted $\mathcal{T}\text{-CL}_{\mathbf{D}_1\mathbf{D}_2\text{-}\mathcal{S}}$, which shows the mapping from $\boldsymbol{\mathcal{T}}_{\mathbf{D}_1\mathbf{D}_2\text{-}\mathcal{S}}$ to $\boldsymbol{\mathcal{T}}_{\mathbf{D}_2\mathbf{D}_1\text{-}\mathcal{S}}$ by conversion (e.g., table 2).

Table 2. Converse list of topological point-line relations in $\mathbb{R}^1$ ($\mathcal{T}\text{-}CL_{\boldsymbol{PL}\text{-}\mathbb{R}^1}$).

r	*disjoint*$_{\mathrm{PL}}$	*meet*$_{\mathrm{PL}}$	*inside*$_{\mathrm{PL}}$
$r^{\smile}$	*disjoint*$_{\mathrm{LP}}$	*meet*$_{\mathrm{LP}}$	*cotains*$_{\mathrm{LP}}$

The composition of a relation r_1 in $\boldsymbol{\mathcal{T}}_{\mathbf{D}_1\mathbf{D}_2\text{-}\mathcal{S}}$ and a relation r_2 in $\boldsymbol{\mathcal{T}}_{\mathbf{D}_2\mathbf{D}_3\text{-}\mathcal{S}}$ is a subset of $\boldsymbol{\mathcal{T}}_{\mathbf{D}_1\mathbf{D}_3\text{-}\mathcal{S}}$. The *composition table* of two topological relation sets $\boldsymbol{\mathcal{T}}_{\mathbf{D}_1\mathbf{D}_2\text{-}\mathcal{S}}$ and $\boldsymbol{\mathcal{T}}_{\mathbf{D}_2\mathbf{D}_3\text{-}\mathcal{S}}$, denoted $\mathcal{T}\text{-CT}_{\mathbf{D}_1\mathbf{D}_2\mathbf{D}_3\text{-}\mathcal{S}}$, shows the mapping from $\boldsymbol{\mathcal{T}}_{\mathbf{D}_1\mathbf{D}_2\text{-}\mathcal{S}} \times \boldsymbol{\mathcal{T}}_{\mathbf{D}_2\mathbf{D}_3\text{-}\mathcal{S}}$ to $\mathbf{2}^{\boldsymbol{\mathcal{T}}_{\mathbf{D}_1\mathbf{D}_3\text{-}\mathcal{S}}}$ (the power-set of $\boldsymbol{\mathcal{T}}_{\mathbf{D}_1\mathbf{D}_3\text{-}\mathcal{S}}$) by composition. In this study, we develop composition tables for the combination of topological relations between simple objects in $\mathbb{R}^1$ (i.e., $\mathcal{T}\text{-CT}_{\mathbf{PPP}\text{-}\mathbb{R}^1}$, $\mathcal{T}\text{-CT}_{\mathbf{PPL}\text{-}\mathbb{R}^1}$, $\mathcal{T}\text{-CT}_{\mathbf{PLP}\text{-}\mathbb{R}^1}$, …, $\mathcal{T}\text{-CT}_{\mathbf{LLL}\text{-}\mathbb{R}^1}$) and for those in $\mathbb{R}^2$ (i.e., $\mathcal{T}\text{-CT}_{\mathbf{PPP}\text{-}\mathbb{R}^2}$, …, $\mathcal{T}\text{-CT}_{\mathbf{RRR}\text{-}\mathbb{R}^2}$).

Given three objects A, B, and C ($A \in \mathbf{D}_A, B \in \mathbf{D}_B, C \in \mathbf{D}_C$), the following set-theoretic constraints, originally introduced in [16] for deriving $\mathcal{T}\text{-CT}_{\mathbf{RRR}\text{-}\mathbb{R}^2}$, always hold for the composition of the topological relation between A and B and that between B and C:

- A's topological part p_A and C's topological part p_C do not intersect if B has a topological part p_B that includes p_A but does not intersect with p_C or that includes p_C but does not intersect with p_A; and
- p_A and p_C intersect if B has a topological part p_B that intersects with p_A and is included in p_C or that intersects with p_C and is included in p_A.

By filtering all relations in $\boldsymbol{\mathcal{T}}_{\mathbf{D}_A\mathbf{D}_C\text{-}\mathcal{S}}$ with these constraints, we obtain the *candidates* for the composition of the topological relation between A and B and that between B and C. Each candidate is examined if they have geometric interpretations. Then, the set of valid candidates are approved as the composition of the topological relation between A and B and that between B and C. By repeating this process for every relation pair in $\boldsymbol{\mathcal{T}}_{\mathbf{D}_A\mathbf{D}_B\text{-}\mathcal{S}} \times \boldsymbol{\mathcal{T}}_{\mathbf{D}_B\mathbf{D}_C\text{-}\mathcal{S}}$, we can develop the composition table of $\boldsymbol{\mathcal{T}}_{\mathbf{D}_A\mathbf{D}_B\text{-}\mathcal{S}}$ and $\boldsymbol{\mathcal{T}}_{\mathbf{D}_B\mathbf{D}_C\text{-}\mathcal{S}}$

(i.e., $\mathcal{T}\text{-CT}_{\mathbf{D}_A\mathbf{D}_B\mathbf{D}_C\text{-}\mathcal{S}}$). For instance, table 3 shows the composition table of line-line relations and line-point relations in $\mathbb{R}^1$ (i.e., $\mathcal{T}\text{-CT}_{\mathbf{LLP}\text{-}\mathbb{R}^1}$) derived by this method.

Table 3. Composition table of topological line-line relations and topological line-point relations in $\mathbb{R}^1$ ($\mathcal{T}\text{-CT}_{\mathbf{LLP}\text{-}\mathbb{R}^1}$).

	disjoint$_{\text{LP}}$	*meet*$_{\text{LP}}$	*contains*$_{\text{LP}}$
equal$_{\text{LL}}$	*disjoint*$_{\text{LP}}$	*meet*$_{\text{LP}}$	*contains*$_{\text{LP}}$
disjoint$_{\text{LL}}$	$\mathbf{U}_{\text{LP}}$	*disjoint*$_{\text{LP}}$	*disjoint*$_{\text{LP}}$
meet$_{\text{LL}}$	$\mathbf{U}_{\text{LP}}$	*disjoint*$_{\text{LP}}$ / *meet*$_{\text{LP}}$	*disjoint*$_{\text{LP}}$
overlaps$_{\text{LL}}$	$\mathbf{U}_{\text{LP}}$	$\mathbf{U}_{\text{LP}}$	$\mathbf{U}_{\text{LP}}$
covers$_{\text{LL}}$	*disjoint*$_{\text{LP}}$	*disjoint*$_{\text{LP}}$ / *meet*$_{\text{LP}}$	$\mathbf{U}_{\text{LP}}$
coveredBy$_{\text{LL}}$	$\mathbf{U}_{\text{LP}}$	*meet*$_{\text{LP}}$ / *contains*$_{\text{LP}}$	*contains*$_{\text{LP}}$
contains$_{\text{LL}}$	*disjoint*$_{\text{LP}}$	*disjoint*$_{\text{LP}}$	$\mathbf{U}_{\text{LP}}$

According to our investigation, the previous two constraints are sufficient when developing the *most* composition tables ($\mathcal{T}\text{-CT}_{\mathbf{PPP}\text{-}\mathbb{R}^1}$, …, $\mathcal{T}\text{-CT}_{\mathbf{LLL}\text{-}\mathbb{R}^1}$ and $\mathcal{T}\text{-CT}_{\mathbf{PPP}\text{-}\mathbb{R}^2}$, …, $\mathcal{T}\text{-CT}_{\mathbf{RRR}\text{-}\mathbb{R}^2}$), but not $\mathcal{T}\text{-CT}_{\mathbf{LLL}\text{-}\mathbb{R}^2}$—in this case, the derived composition candidates may have no geometric interpretation. For instance, imagine that there are three simple lines A, B, and C, where A contains B and B crosses C (figure 2). Obviously, A cannot contain C. The constraints in equation 4, however, do not exclude the composition candidate where A contains C.

Figure 2. The arrangements of three lines A, B, and C, from which we can conclude that A cannot contain C.

In general, when A contains/covers B, A cannot contains/covers C if:

- C directly links B's interior and exterior (figure 3);
- B directly links C's interior and exterior; or
- A covers B and B is inside of C (figure 4).

We can tell from the given relations that the first two situations occur whenever the relation between B and C belong to the topological line-line relations not realizable in $\mathbb{R}^1$ (i.e., the line-line relations other than *equal*$_{\text{LL}}$, *disjoint*$_{\text{LL}}$, *meet*$_{\text{LL}}$, *overlaps*$_{\text{LL}}$, *covers*$_{\text{LL}}$, *contains*$_{\text{LL}}$, and *inside*$_{\text{LL}}$). Thus, the previous condition is simplified as follows:

- If $contains_{\text{LL}}(A, B)$ and the relation between B and C is neither *equal*$_{\text{LL}}$, *disjoint*$_{\text{LL}}$, *meet*$_{\text{LL}}$, *overlaps*$_{\text{LL}}$, *covers*$_{\text{LL}}$, *contains*$_{\text{LL}}$, nor *inside*$_{\text{LL}}$, then $A^- \cap C^\circ = \neg\emptyset$;
- If $covers_{\text{LL}}(A, B)$ and the relation between B and C is neither *equal*$_{\text{LL}}$, *disjoint*$_{\text{LL}}$, *meet*$_{\text{LL}}$, *overlaps*$_{\text{LL}}$, *covers*$_{\text{LL}}$, nor *contains*$_{\text{LL}}$, then $A^- \cap C^\circ = \neg\emptyset$;

Similarly, the following two constraints hold:

- If $inside_{\text{LL}}(B, C)$ and the relation between A and B is neither *equal*$_{\text{LL}}$, *disjoint*$_{\text{LL}}$, *meet*$_{\text{LL}}$, *overlaps*$_{\text{LL}}$, *covers*$_{\text{LL}}$, *contains*$_{\text{LL}}$, nor *inside*$_{\text{LL}}$, then $A^\circ \cap C^- = \neg\emptyset$; and
- If $coveredBy_{\text{LL}}(B, C)$ and the relation between A and B is neither *equal*$_{\text{LL}}$, *disjoint*$_{\text{LL}}$, *meet*$_{\text{LL}}$, *overlaps*$_{\text{LL}}$, *covers*$_{\text{LL}}$, nor *contains*$_{\text{LL}}$, then $A^\circ \cap C^- = \neg\emptyset$.

By adding these four constraints, we can successfully derive the composition candidates that have geometric interpretations and, accordingly, we can develop the composition table accordingly, we can develop the composition table $\mathcal{T}\text{-CT}_{\mathbf{LLL}\text{-}\mathbb{R}^2}$.

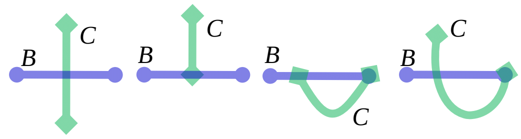


Figure 3. Examples of arrangements where a line C directly connects the interior and exterior of a line B.

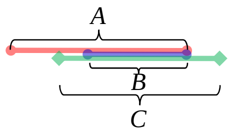


Figure 4. If a line A covers a line B and B is inside of a line C, then A cannot contain/covers C.

5.9-Intersection Calculi

First, to conduct reasoning on topological relations between single-type objects, we introduce *homogeneous 9-intersection calculi*. These calculi are defined for each object domain and each space. The homogeneous 9-intersection calculus for the object domain $\mathbf{D}$ and the space $\mathcal{S}$, denoted $\text{Homo9IC}_{\mathbf{D}\text{-}\mathcal{S}}$, is formulated based on the following elements:

- a set of base relations $\boldsymbol{\mathcal{T}}_{\mathbf{DD}\text{-}\mathcal{S}}$;
- a converse list $\mathcal{T}\text{-CL}_{\mathbf{DD}\text{-}\mathcal{S}}$; and
- a composition table $\mathcal{T}\text{-CT}_{\mathbf{DDD}\text{-}\mathcal{S}}$.

These elements satisfy the requirements of ordinary qualitative spatial calculi (non-associative algebra); that is,

- $\boldsymbol{\mathcal{T}}_{\mathbf{DD}\text{-}\mathcal{S}}$ is jointly exclusive, pairwise disjoint, and contains an identity element;
- the converse operation on $\boldsymbol{\mathcal{T}}_{\mathbf{DD}\text{-}\mathcal{S}}$ is closed under $\boldsymbol{\mathcal{T}}_{\mathbf{DD}\text{-}\mathcal{S}}$ (i.e., $\forall r \in \boldsymbol{\mathcal{T}}_{\mathbf{DD}\text{-}\mathcal{S}}\ r^{\cup} \in \boldsymbol{\mathcal{T}}_{\mathbf{DD}\text{-}\mathcal{S}}$); and
- the composition operation on $\boldsymbol{\mathcal{T}}_{\mathbf{DD}\text{-}\mathcal{S}}$ is closed under $\boldsymbol{\mathcal{T}}_{\mathbf{DD}\text{-}\mathcal{S}}$ (i.e., $\forall r_1, r_2 \in \boldsymbol{\mathcal{T}}_{\mathbf{DD}\text{-}\mathcal{S}}\ r_1; r_2 \subseteq \boldsymbol{\mathcal{T}}_{\mathbf{DD}\text{-}\mathcal{S}}$).

Consequently, spatial reasoning can be conducted in an algebraic framework.

Next, to conduct reasoning on topological relations between various combinations of objects, *heterogeneous 9-intersection calculi* are introduced. These calculi are developed for each space. The heterogeneous 9-intersection calculus for the space $\mathcal{S}$, denoted Het9IC$_{\mathcal{S}}$, have the ability to deal with all sorts of simple objects (points, simple lines, simple regions, and simple bodies) that $\mathcal{S}$ can contain. Naturally, if $\mathcal{S}$ is a d-dimension space, Het9IC$_{\mathcal{S}}$ covers:

- d+1 object domains $\{\mathbf{D}_0, \ldots, \mathbf{D}_d\}$;
- $(d+1)^2$ sets of topological relations $\left\{\boldsymbol{\mathcal{T}}_{\mathbf{D}_i\mathbf{D}_j\text{-}\mathcal{S}}\right\}_{i,j\in\{0,\ldots,d\}}$;
- $(d+1)^2$ converse lists $\left\{\mathcal{T}\text{-CL}_{\mathbf{D}_i\mathbf{D}_j\text{-}\mathcal{S}}\right\}_{i,j\in\{0,\ldots,d\}}$; and
- $(d+1)^3$ composition tables $\left\{\mathcal{T}\text{-CT}_{\mathbf{D}_i\mathbf{D}_j\mathbf{D}_k\text{-}\mathcal{S}}\right\}_{i,j,k\in\{0,\ldots,d\}}$

For instance, Het9IC$_{\mathbb{R}^1}$ covers:

- two object domains: **P** and **L**;
- four sets of topological relations: $\boldsymbol{\mathcal{T}}_{\mathbf{PP}\text{-}\mathbb{R}^1}$, $\boldsymbol{\mathcal{T}}_{\mathbf{PL}\text{-}\mathbb{R}^1}$, $\boldsymbol{\mathcal{T}}_{\mathbf{LP}\text{-}\mathbb{R}^1}$, and $\boldsymbol{\mathcal{T}}_{\mathbf{LL}\text{-}\mathbb{R}^1}$;
- four converse lists: $\mathcal{T}\text{-CL}_{\mathbf{PP}\text{-}\mathbb{R}^1}$, $\mathcal{T}\text{-CL}_{\mathbf{PL}\text{-}\mathbb{R}^1}$, $\mathcal{T}\text{-CL}_{\mathbf{LP}\text{-}\mathbb{R}^1}$, and $\mathcal{T}\text{-CL}_{\mathbf{LL}\text{-}\mathbb{R}^1}$; and
- eight composition tables: $\mathcal{T}\text{-CT}_{\mathbf{PPP}\text{-}\mathbb{R}^1}$, $\mathcal{T}\text{-CT}_{\mathbf{PPL}\text{-}\mathbb{R}^1}$, $\mathcal{T}\text{-CT}_{\mathbf{PLP}\text{-}\mathbb{R}^1}$, $\mathcal{T}\text{-CT}_{\mathbf{PLL}\text{-}\mathbb{R}^1}$, …, $\mathcal{T}\text{-CT}_{\mathbf{LLL}\text{-}\mathbb{R}^1}$.

These elements are integrated as follows. First, the *generalized object domain* $\mathbf{D}^*$ and the generalized base relations $\boldsymbol{\mathcal{B}}^*$ are defined as follows:

- $\mathbf{D}^* = \bigcup_{i\in\{1,\ldots,d\}} \mathbf{D}_i$
- $\boldsymbol{\mathcal{B}}^* = \left(\bigcup_{i,j\in\{1,\ldots,d\}} \boldsymbol{\mathcal{B}}_{\mathbf{D}_i\mathbf{D}_j}\right) \cup \{equal\}$

Basically $\boldsymbol{\mathcal{B}}^*$ refers to the relations between two arbitrary objects in $\mathbf{D}^*$, but it also contains an identity relation ($equal$). The presence of an identity relation is a requirement of the calculi's algebraic framework (non-associative algebra). This identity element is different from domain-level identity elements $equal_{\mathbf{D}_i\mathbf{D}_i}$. In Het9IC$_{\mathbb{R}^1}$, for instance, $\boldsymbol{\mathcal{B}}^*$ contains $equal$, $equal_{\mathbf{PP}}$, and $equal_{\mathbf{LL}}$. We did not integrate these identity elements to prevent senseless compositions. For instance, $equal_{\mathbf{PP}}; disjoint_{\mathbf{LL}}$ must be empty since the composition of a point-point relation and a line-line relation is impossible. However, $equal; disjoint_{\mathbf{LL}} = \{disjoint_{\mathbf{LL}}\}$ by definition and, accordingly, it is not appropriate to substitute $equal_{\mathbf{PP}}$ by $equal$. Thus, $equal$ is considered a purely abstract relation with no geometric interpretation (i.e., $\forall A, B \in \mathbf{D}^*\ (A, B) \notin equal$). Then, we can consider $\boldsymbol{\mathcal{B}}^*$ a jointly exhaustive and pairwise disjoint set of base relations, which is also a requirement of the calculi's algebraic framework.

Next, we integrate the relevant converse lists and composition tables. The integrated converse list $\mathcal{T}\text{-CL}_{\mathbf{D}^*\mathbf{D}^*\text{-}\mathcal{S}}$ is derived by concatenating the relevant converse lists $\left\{\mathcal{T}\text{-CL}_{\mathbf{D}_i\mathbf{D}_j\text{-}\mathcal{S}}\right\}$ and adding an item "$equal^{\cup} = equal$." For instance, table 4 shows $\mathcal{T}\text{-CL}_{\mathbf{D}^*\mathbf{D}^*\text{-}\mathbb{R}^1}$, which is derived from $\mathcal{T}\text{-CL}_{\mathbf{PP}\text{-}\mathbb{R}^1}$, ..., $\mathcal{T}\text{-CL}_{\mathbf{LL}\text{-}\mathbb{R}^1}$. Similarly, the integrated composition table $\mathcal{T}\text{-CT}_{\mathbf{D}^*\mathbf{D}^*\mathbf{D}^*\text{-}\mathcal{S}}$ is derived by adjoining the relevant composition tables $\left\{\mathcal{T}\text{-CT}_{\mathbf{D}_i\mathbf{D}_j\mathbf{D}_k\text{-}\mathcal{S}}\right\}$ and adding one row and one column about $equal$-related composition. For instance, table 5 shows $\mathcal{T}\text{-CT}_{\mathbf{D}^*\mathbf{D}^*\mathbf{D}^*\text{-}\mathbb{R}^1}$, which is derived from $\mathcal{T}\text{-CT}_{\mathbf{PPP}\text{-}\mathbb{R}^1}$, …, $\mathcal{T}\text{-CT}_{\mathbf{LLL}\text{-}\mathbb{R}^1}$.

Table 4. Integrated converse list $\mathcal{T}\text{-}CL_{\boldsymbol{D}^*\boldsymbol{D}^*\text{-}\mathbb{R}^1}$ (the highlighted part corresponds to $\mathcal{T}\text{-}CL_{\boldsymbol{PL}\text{-}\mathbb{R}^1}$ in Table 2).

r	$r^{\cup}$	r	$r^{\cup}$
equal	*equal*	$equal_{\mathrm{LL}}$	$equal_{\mathrm{LL}}$
$equal_{\mathrm{PP}}$	$equal_{\mathrm{PP}}$	$disjoint_{\mathrm{LL}}$	$disjoint_{\mathrm{LL}}$
$disjoint_{\mathrm{PP}}$	$disjoint_{\mathrm{PP}}$	$meet_{\mathrm{LL}}$	$meet_{\mathrm{LL}}$
$disjoint_{\mathrm{PL}}$	$disjoint_{\mathrm{LP}}$	$overlaps_{\mathrm{LL}}$	$overlaps_{\mathrm{LL}}$
$meet_{\mathrm{PL}}$	$meet_{\mathrm{LP}}$	$covers_{\mathrm{LL}}$	$coveredBy_{\mathrm{LL}}$
$inside_{\mathrm{PL}}$	$contains_{\mathrm{LP}}$	$coveredBy_{\mathrm{LL}}$	$covers_{\mathrm{LL}}$
$disjoint_{\mathrm{LP}}$	$disjoint_{\mathrm{PL}}$	$contains_{\mathrm{LL}}$	$inside_{\mathrm{LL}}$
$meet_{\mathrm{LP}}$	$meet_{\mathrm{PL}}$	$inside_{\mathrm{LL}}$	$contains_{\mathrm{LL}}$
$contains_{\mathrm{LP}}$	$inside_{\mathrm{PL}}$		

Now we have:

- an integrated set of base relations $\boldsymbol{\mathcal{B}}^*$, which is jointly exclusive, pairwise disjoint, and contains an identity relation;
- an integrated converse list $\mathcal{TR}\text{-CL}_{\mathbf{D}^*\mathbf{D}^*\text{-}\mathbb{R}^1}$, which is closed under $\boldsymbol{\mathcal{B}}^*$; and
- an integrated composition table $\mathcal{T}\text{-CT}_{\mathbf{D}^*\mathbf{D}^*\mathbf{D}^*\text{-}\mathbb{R}^1}$, which is closed under $\boldsymbol{\mathcal{B}}^*$.

Consequently, it is expected that spatial reasoning on the topological relations between arbitrary objects in $\mathbf{D}^*$ can be conducted in an algebraic framework, just like we can do in ordinary single-domain spatial calculi. This will be demonstrated in Section 6.

Simple Assessment of 9-Intersection Calculi

Based on the converse lists and composition tables developed in Section 4, we developed seven basic calculi: Homo9IC$_{\mathbf{P}\text{-}\mathbb{R}^1}$, Homo9IC$_{\mathbf{L}\text{-}\mathbb{R}^1}$, Homo9IC$_{\mathbf{P}\text{-}\mathbb{R}^2}$, Homo9IC$_{\mathbf{L}\text{-}\mathbb{R}^2}$, Homo9IC$_{\mathbf{R}\text{-}\mathbb{R}^2}$, Het9IC$_{\mathbb{R}^1}$, and Het9IC$_{\mathbb{R}^2}$. We conducted simple assessment of these calculi.

First, for the composition table of each calculus, we calculated the *crispness* and the *ratio of unique compositions* (table 5). These two measures are used in spatial database studies for assessing the effectiveness of composition tables [17, 18]. We found that Het9IC$_{\mathbb{R}^1}$ and Het9IC$_{\mathbb{R}^2}$ marked high crispness, but this result is not so meaningful because the integration of composition tables yields the increase of relations not contained in each composition and increases the crispness. We also found that Homo9IC$_{\mathbf{L}\text{-}\mathbb{R}^2}$'s ratio of unique compositions was very low. This is because in many compositions the presence or absence of intersection between two lines' interiors cannot be determined. Het9IC$_{\mathbb{R}^2}$'s ratio of unique compositions was also low, because its composition table has many empty cells that correspond to impossible compositions.

Table 5. Integrated composition table $\mathcal{T}\text{-}CT_{\boldsymbol{D^*D^*D^*}\text{-}\mathbb{R}^1}$ (the highlighted part corresponds to $\mathcal{T}\text{-}CT_{\boldsymbol{LLP}\text{-}\mathbb{R}^1}$ in Table 3;
eq: *equal*, *dj*: *disjoint*, *mt*: *meet*, *ov*: *overlap*, *cv*: *covers*, *cB*: *coveredBy*, *ct*: *contains*, and *in*:*inside*).

	eq	eq_{PP}	dj_{PP}	dj_{PL}	mt_{PL}	in_{PL}	dj_{LP}	mt_{LP}	ct_{LP}	eq_{LL}	dj_{LL}	mt_{LL}	ov_{LL}	cv_{LL}	cB_{LL}	ct_{LL}	in_{LL}
eq	eq	eq_{PP}	dj_{PP}	dj_{PL}	mt_{PL}	in_{PL}	dj_{LP}	mt_{LP}	ct_{LP}	eq_{LL}	dj_{LL}	mt_{LL}	ov_{LL}	cv_{LL}	cB_{LL}	ct_{LL}	in_{LL}
eq_{PP}	eq_{PP}	eq_{PP}	dj_{PP}	dj_{PL}	mt_{PL}	in_{PL}											
dj_{PP}	dj_{PP}	dj_{PP}	dj_{PP}	$\mathbf{U}_{PL}$	$\mathbf{U}_{PL}$	$\mathbf{U}_{PL}$											
dj_{PL}	dj_{PL}						$\mathbf{U}_{PP}$	dj_{PP}	dj_{PP}	dj_{PL}	$\mathbf{U}_{PL}$	$\mathbf{U}_{PL}$	$\mathbf{U}_{PL}$	dj_{PL}	$\mathbf{U}_{PL}$	dj_{PL}	$\mathbf{U}_{PL}$
mt_{PL}	mt_{PL}						dj_{PP}	$\mathbf{U}_{PP}$	dj_{PP}	mt_{PL}	dj_{PL}	dj_{PL} mt_{PL}	$\mathbf{U}_{PL}$	dj_{PL} mt_{PL}	mt_{PL} in_{PL}	dj_{PL}	in_{PL}
in_{PL}	in_{PL}						dj_{PP}	dj_{PP}	$\mathbf{U}_{PP}$	in_{PL}	dj_{PL}	dj_{PL}	$\mathbf{U}_{PL}$	$\mathbf{U}_{PL}$	in_{PL}	$\mathbf{U}_{PL}$	in_{PL}
dj_{LP}	dj_{LP}	dj_{LP}	$\mathbf{U}_{PL}$	$\mathbf{U}_{LL}$	dj_{LL} mt_{LL} ov_{LL} cB_{LL} in_{LL}	dj_{LL} mt_{LL} ov_{LL} cB_{LL} in_{LL}											
mt_{LP}	mt_{LP}	mt_{LP}	$\mathbf{U}_{PL}$	dj_{LL} mt_{LL} ov_{LL} cB_{LL} in_{LL}	eq_{LL} mt_{LL} ov_{LL} cv_{LL} cB_{LL}	ov_{LL} cB_{LL} in_{LL}											
ct_{LP}	ct_{LP}	ct_{LP}	$\mathbf{U}_{PL}$	dj_{LL} mt_{LL} ov_{LL} cB_{LL} in_{LL}	ov_{LL} cv_{LL} ct_{LL}	ov_{LL} cv_{LL} cB_{LL} ct_{LL} in_{LL}											
eq_{LL}	eq_{LL}						dj_{LP}	mt_{LP}	ct_{LP}	eq_{LL}	dj_{LL}	mt_{LL}	ov_{LL}	cv_{LL}	cB_{LL}	ct_{LL}	in_{LL}
dj_{LL}	dj_{LL}						$\mathbf{U}_{LP}$	dj_{LP}	dj_{LP}	dj_{LL}	$\mathbf{U}_{LL}$	dj_{LL} mt_{LL} ov_{LL} cB_{LL} in_{LL}	dj_{LL} mt_{LL} ov_{LL} cB_{LL} in_{LL}	dj_{LL}	dj_{LL} mt_{LL} ov_{LL} cB_{LL} in_{LL}	dj_{LL}	dj_{LL} mt_{LL} ov_{LL} cB_{LL} in_{LL}
mt_{LL}	mt_{LL}						$\mathbf{U}_{LP}$	dj_{LP} mt_{LP}	dj_{LP}	mt_{LL}	dj_{LL} mt_{LL} ov_{LL} cv_{LL} ct_{LL}	eq_{LL} dj_{LL} mt_{LL} ov_{LL} cv_{LL} cB_{LL}	dj_{LL} mt_{LL} ov_{LL} cB_{LL} in_{LL}	dj_{LL} mt_{LL}	mt_{LL} ov_{LL} cB_{LL} in_{LL}	dj_{LL}	ov_{LL} cB_{LL} in_{LL}
ov_{LL}	ov_{LL}						$\mathbf{U}_{LP}$	$\mathbf{U}_{LP}$	$\mathbf{U}_{LP}$	ov_{LL}	dj_{LL} mt_{LL} ov_{LL} cv_{LL} ct_{LL}	dj_{LL} mt_{LL} ov_{LL} cv_{LL} ct_{LL}	$\mathbf{U}_{LL}$	dj_{LL} mt_{LL} ov_{LL} cv_{LL} ct_{LL}	ov_{LL} cB_{LL} in_{LL}	dj_{LL} mt_{LL} ov_{LL} cv_{LL} ct_{LL}	ov_{LL} cB_{LL} in_{LL}
cv_{LL}	cv_{LL}						$\mathbf{U}_{LP}$	mt_{LP} ct_{LP}	ct_{LP}	cv_{LL}	dj_{LL} mt_{LL} ov_{LL} cv_{LL} ct_{LL}	mt_{LL} ov_{LL} cv_{LL} ct_{LL}	ov_{LL} cv_{LL} ct_{LL}	cv_{LL} ct_{LL}	eq_{LL} ov_{LL} cv_{LL} cB_{LL}	ct_{LL}	ov_{LL} cB_{LL} in_{LL}
cB_{LL}	cB_{LL}						dj_{LP}	dj_{LP} mt_{LP}	$\mathbf{U}_{LP}$	cB_{LL}	dj_{LL}	dj_{LL} mt_{LL}	dj_{LL} mt_{LL} ov_{LL} cB_{LL} in_{LL}	eq_{LL} dj_{LL} mt_{LL} ov_{LL} cv_{LL} cB_{LL}	cB_{LL} in_{LL}	dj_{LL} mt_{LL} ov_{LL} cv_{LL} ct_{LL}	in_{LL}
ct_{LL}	ct_{LL}						$\mathbf{U}_{LP}$	ct_{LP}	ct_{LP}	ct_{LL}	dj_{LL} mt_{LL} ov_{LL} cv_{LL} ct_{LL}	ov_{LL} cv_{LL} ct_{LL}	ov_{LL} cv_{LL} ct_{LL}	ct_{LL}	ov_{LL} cv_{LL} ct_{LL}	ct_{LL}	eq_{LL} ov_{LL} cv_{LL} cB_{LL} ct_{LL} in_{LL}
in_{LL}	in_{LL}						dj_{LP}	dj_{LP}	$\mathbf{U}_{LP}$	in_{LL}	dj_{LL}	dj_{LL}	dj_{LL} mt_{LL} ov_{LL} cB_{LL} in_{LL}	dj_{LL} mt_{LL} ov_{LL} cB_{LL} in_{LL}	in_{LL}	$\mathbf{U}_{LL}$	in_{LL}

Next, we examined the associativity of the compositions in each calculus (i.e., whether $(\mathrm{A};\mathrm{B});\mathrm{C}=\mathrm{A};(\mathrm{B};\mathrm{C})$ holds or not) (table 6). We found that $\mathrm{Homo9IC}_{\mathbf{L}\text{-}\mathbb{R}^2}$ is not associative, and accordingly $\mathrm{Het9IC}_{\mathbb{R}^2}$ as well. Alternatively, the compositions in these two calculi satisfy semi-associativity (i.e., $(\mathrm{A};\mathbf{U});\mathbf{U}=\mathrm{A};(\mathbf{U};\mathbf{U})$ holds). On the other hand, other five calculi are all associative. One example of non-associativity in $\mathrm{Homo9IC}_{\mathbf{L}\text{-}\mathbb{R}^2}$ is that $covers_{\mathrm{LL}};(covers_{\mathrm{LL}};cross_{\mathrm{LL}})\ni covers_{\mathrm{LL}}$ (equations 4-6), but $(covers_{\mathrm{LL}};covers_{\mathrm{LL}});cross_{\mathrm{LL}}\not\ni covers_{\mathrm{LL}}$ (equations 7-10). This conflict arises from the ambiguity of the pattern $\begin{pmatrix}\neg\emptyset & \neg\emptyset & \neg\emptyset\\ \neg\emptyset & \emptyset & \neg\emptyset\\ \neg\emptyset & \neg\emptyset & \neg\emptyset\end{pmatrix}$. In equation 4, this pattern is interpreted as *diverge&cross&divergedBy*$_{\mathrm{LL}}$ relation (see *B* and *D* in figure 5a), whereas in equation 6 this pattern is interpreted as *overlap*$_{\mathrm{LL}}$ relation (see *B* and *D* in figure 5b). Unfortunately, the ambiguity of this pattern is an intrinsic problem of the 9-intersection.

$$covers_{\mathrm{LL}};cross_{\mathrm{LL}}\ni\begin{pmatrix}\neg\emptyset & \neg\emptyset & \neg\emptyset\\ \neg\emptyset & \emptyset & \neg\emptyset\\ \neg\emptyset & \neg\emptyset & \neg\emptyset\end{pmatrix}_{\mathrm{LL}} \quad (4)$$

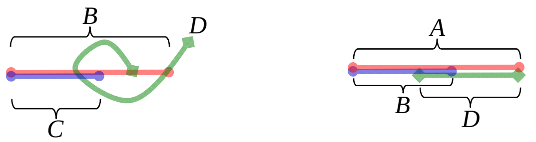
$$covers_{\mathrm{LL}};\begin{pmatrix}\neg\emptyset & \neg\emptyset & \neg\emptyset\\ \neg\emptyset & \emptyset & \neg\emptyset\\ \neg\emptyset & \neg\emptyset & \neg\emptyset\end{pmatrix}_{\mathrm{LL}}\ni covers_{\mathrm{LL}} \quad (5)$$

$$\therefore covers_{\mathrm{LL}};(covers_{\mathrm{LL}};cross_{\mathrm{LL}})\ni covers_{\mathrm{LL}} \quad (6)$$

$$\begin{aligned}&(covers_{\mathrm{LL}};covers_{\mathrm{LL}});cross_{\mathrm{LL}}\\&=(covers_{\mathrm{LL}}\vee contains_{\mathrm{LL}});cross_{\mathrm{LL}} &(7)\\&=covers_{\mathrm{LL}};cross_{\mathrm{LL}}\vee contains_{\mathrm{LL}};cross_{\mathrm{LL}}\\&covers_{\mathrm{LL}};cross_{\mathrm{LL}}\not\ni covers_{\mathrm{LL}} &(8)\\&contains_{\mathrm{LL}};cross_{\mathrm{LL}}\not\ni covers_{\mathrm{LL}} &(9)\\&\therefore(covers_{\mathrm{LL}};covers_{\mathrm{LL}});cross_{\mathrm{LL}}\not\ni covers_{\mathrm{LL}} &(10)\end{aligned}$$

Table 6. Properties of 9-intersection calculi for $\mathbb{R}^1$ and $\mathbb{R}^2$.

	$\mathrm{Homo9IC}_{\mathbf{P}\text{-}\mathbb{R}^1}$	$\mathrm{Homo9IC}_{\mathbf{L}\text{-}\mathbb{R}^1}$	$\mathrm{Homo9IC}_{\mathbf{P}\text{-}\mathbb{R}^2}$	$\mathrm{Homo9IC}_{\mathbf{L}\text{-}\mathbb{R}^2}$	$\mathrm{Homo9IC}_{\mathbf{R}\text{-}\mathbb{R}^2}$	$\mathrm{Het9IC}_{\mathbb{R}^1}$,	$\mathrm{Het9IC}_{\mathbb{R}^2}$
Number of relations	2	8	2	33	8	17	94
Crispness of compositions	.375	.623	.375	.645	.623	.869	.910
Ratio of unique composition	.750	.422	.750	.080	.422	.553	.167
Associativity	√	√	√		√	√	
Semi-Associativity	√	√	√	√	√	√	√



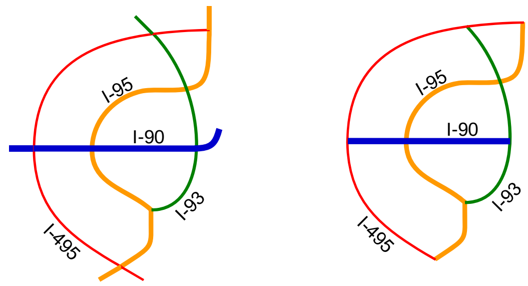
(*a*) (*b*)

Figure 5. Illustrations of (a) equation 5 and (b) equation 6.

6. Examples

This section demonstrates the application of the proposed 9-intersection calculi for qualitative spatial reasoning. We start from $\mathrm{Homo9IC}_{\mathbf{L}\text{-}\mathbb{R}^2}$ as the representative of homogeneous 9-intersection calculi.

In the Boston metropolitan area, there are four interstate highways; I-90, I-93, I-95, and I-495. Their actual network is like figure 6a, but here we consider a simplified network in figure 6b. Table 7 lists the topological relations between the highways in the simplified network. Imagine that we drive two of these four highways and observe their connections to other highways. For instance, figure 7a/b illustrates the knowledge obtained from the drive on I-90 and I-95/I-93. Based on such knowledge, what can we say about the relations between the remaining two highways? With $\mathrm{Homo9IC}_{\mathbf{L}\text{-}\mathbb{R}^2}$, we can derive possible relations between unvisited highways from partial knowledge about the highway network.

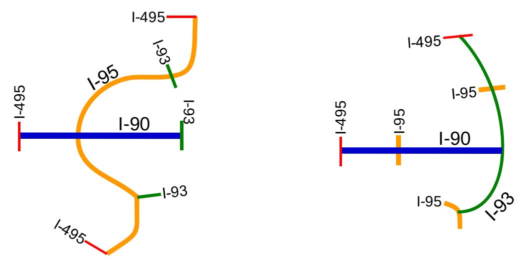


(*a*) (*b*)

Figure 6. (a) Network of four interstate highways in the Boston metropolitan area and (b) its simplified version for experiment.

Table 7. Topological relations between pairs of highways in the network of figure 6b.

	I-90	I-93	I-95	I-495
I-90	–	*diverges*$_{\mathrm{LL}}$	*cross*$_{\mathrm{LL}}$	*diverges*$_{\mathrm{LL}}$
I-93	*divergedBy*$_{\mathrm{LL}}$	–	*diverges &cross*$_{\mathrm{LL}}$	*diverges*$_{\mathrm{LL}}$
I-95	*cross*$_{\mathrm{LL}}$	*divergedBy &cross*$_{\mathrm{LL}}$	–	*meet-at-both-ends*$_{\mathrm{LL}}$
I-495	*divergedBy*$_{\mathrm{LL}}$	*divergedBy*$_{\mathrm{LL}}$	*meet-at-both-ends*$_{\mathrm{LL}}$	–



(*a*) (*b*)

Figure 7. Partial knowledge about the highway network, obtained through the drive on (a) I-90 and I-95 and (b)I-90 and I-93.

For actual computation, we put the data of Homo9IC$_{\mathbf{L}\text{-}\mathbb{R}^2}$ ($\boldsymbol{\mathcal{T}}_{\mathbf{LL}\text{-}\mathbb{R}^2}$, $\mathcal{T}$-CL$_{\mathbf{LL}\text{-}\mathbb{R}^2}$, and $\mathcal{T}$-CT$_{\mathbf{LLL}\text{-}\mathbb{R}^2}$) into SparQ [8] and calculate all consistent scenarios under the constraint network that follows table 7 but replaces the relations between unvisited highways by $\mathbf{U}_{\mathrm{LL}}$. The computation result is shown in table 8. For each pair of unvisited highways we obtained 4 to 28 possible relations. Each solution successfully contained the actual relation in the network of figure 6b.

Table 8. Possible relations between pairs of unvisited highways derived as algebraically-consistent scenarios.

Unknown relation	Derived solution (dv: diverges, dB: divergedBy)
(I95, I495)	All but equal$_{\mathrm{LL}}$, *covers*$_{\mathrm{LL}}$, *coveredBy*$_{\mathrm{LL}}$, *contains*$_{\mathrm{LL}}$, *inside*$_{\mathrm{LL}}$
(I93, I495)	*disjoint*$_{\mathrm{LL}}$, *cross*$_{\mathrm{LL}}$, ***diverges***$_{\mathrm{LL}}$, *diverges&cross*$_{\mathrm{LL}}$
(I93, I95)	*disjoint*$_{\mathrm{LL}}$, *cross*$_{\mathrm{LL}}$, *diverges*$_{\mathrm{LL}}$, ***diverges&cross***$_{\mathrm{LL}}$
(I90, I495)	*disjoint*$_{\mathrm{LL}}$, *cross*$_{\mathrm{LL}}$, ***diverges***$_{\mathrm{LL}}$, *diverges&cross*$_{\mathrm{LL}}$
(I90, I95)	*disjoint*$_{\mathrm{LL}}$, ***cross***$_{\mathrm{LL}}$, *diverges*$_{\mathrm{LL}}$, *diverges&cross*$_{\mathrm{LL}}$
(I90, I93)	*disjoint*$_{\mathrm{LL}}$, *covers*$_{\mathrm{LL}}$, *coveredBy*$_{\mathrm{LL}}$, *cross*$_{\mathrm{LL}}$, *meet*$_{\mathrm{LL}}$, *meet&cross*$_{\mathrm{LL}}$, ***diverges***$_{\mathrm{LL}}$, *dv&cross*$_{\mathrm{LL}}$, *divergedBy*$_{\mathrm{LL}}$, *dB&cross*$_{\mathrm{LL}}$, *dv&meet*$_{\mathrm{LL}}$, *dv&cross&meet*$_{\mathrm{LL}}$, *dB&meet*$_{\mathrm{LL}}$, *dB&cross&meet*$_{\mathrm{LL}}$, *dv&dB*$_{\mathrm{LL}}$, *dv&cross&dB*$_{\mathrm{LL}}$, *dv&dB&meet*$_{\mathrm{LL}}$, *dv&cross&dB&meet*$_{\mathrm{LL}}$

For the scenario in figure 7a, we obtained four possible relations between I-93 and I-495—*disjoint*$_{\mathrm{LL}}$, *cross*$_{\mathrm{LL}}$, *diverges*$_{\mathrm{LL}}$, and *diverges&cross*$_{\mathrm{LL}}$. This solution look reasonable, since we can say from the given knowledge that (i) both endpoints of I-495 do not intersect with I-93 and (ii) one endpoint of I-93 do not intersect with I-495. On the other hand, for the scenario in figure 7b, we obtained as many as 28 possible relations between I-95 and I-495, because from the given knowledge we can only say that I-95 is neither contained nor covered by I-495 and vice versa.

Next, we enrich the previous map by adding two areas—Boston city and the urban area (covering the Boston city). Their spatial arrangement is illustrated in figure 8. Imagine that we have driven three of the four highways and obtained the knowledge about how the three highways connect to other highways and two areas. Based on such knowledge, what can we say about the relation between the remaining highway and two districts? For instance, in figure 9a/b, how I-95/I-93 goes with regard to two districts? To solve this problem, we use Het9IC$_{\mathbb{R}^2}$, since it concerns the following heterogeneous sets of topological relations:

- topological line-line relations between the visited highways;
- topological line-region between the visited highways and the two areas; and
- topological region-region relations between the two areas—$\mathit{coveredBy}_{\mathrm{RR}}(\mathit{Boston}, \mathit{Urban})$.

In addition, we also use the following optional information to obtain finer solutions:

- topological point-region relations between the highway junctions and the two areas; and
- topological line-point relations between the visited highways and the highway junctions.

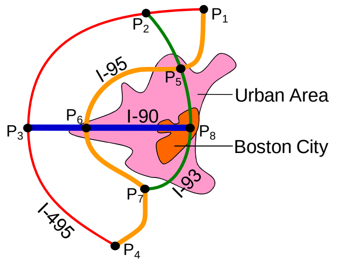
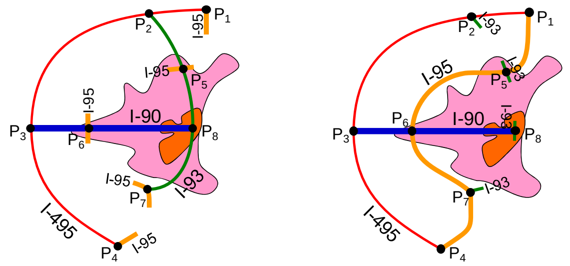


Figure 8. Spatial arrangement of highways and two districts in the Boston metropolitan area.



(*a*) (*b*)

Figure 9. Partial knowledge about the highway network, obtained through the drive on highways except (a) I-95 and (b) I-93.

For actual computation, we put the data of Het9IC$_{\mathbb{R}^2}$ into SparQ and calculated all consistent scenarios under the constraints described above. We obtained 1 to 16 possible relations for each scenario (table 9). Each solution successfully contains the actual relations in the highway-district arrangement of figure 7a.

For the scenario in figure 9a, we obtained a solution in which I-95 goes through the urban area, and either goes through, touches, or avoids the Boston city. This solution looks reasonable, as we know that (i) both endpoints of I-95 are out of the two districts and (ii) I-95 passes through the urban area. The solutions for I-90 and I-495 also look reasonable. The solution for I-93, however, looks strange. Even though both endpoints of I-93 are located out of the urban area (figure 9b), the derived solution does not filter out such unrealizable relations as $\mathit{goInto}_{\mathrm{LR}}(\text{I-93}, \mathrm{Urban})$. This is because the current reasoning process does not use the commonsense knowledge that the line's boundary consists of two endpoints, but regard it simply as a point set[1]

[1] On the other hand, in figure 9a, the possibility of $\mathit{goInto}_{\mathrm{LR}}(\text{I-95}, \mathrm{Urban})$ is successfully excluded, because the data already tells that I-95's boundary is completely contained in I-495's interior.

This indicates that we can still improve the reasoning making use of the structural information of spatial objects.

Table 9. Possible relations between an unvisited highway and two districts, derived as algebraically-consistent scenarios.

Unknown relations	Derived solution (gT_{LR}: *goThrough*$_{LR}$, $gIBE_{LR}$: *goIntoThenBackToEdge*$_{LR}$)
[(I-495, Urban), (I-495, Boston)]	[gT_{LR}, gT_{LR}], [gT_{LR}, *touch*$_{LR}$], [gT_{LR}, *disjoint*$_{LR}$], [*touch*$_{LR}$, *touch*$_{LR}$], [*touch*$_{LR}$, *disjoint*$_{LR}$], **[*disjoint*$_{LR}$, *disjoint*$_{LR}$]**
[(I-95, Urban), (I-95, Boston)]	[gT_{LR}, gT_{LR}], [gT_{LR}, *touch*$_{LR}$], **[gT_{LR}, *disjoint*$_{LR}$]**
[(I-93, Urban), (I-93, Boston)]	**[gT_{LR}, gT_{LR}]**, [*goInto*$_{LR}$, *goInto*$_{LR}$], [*goInto*$_{LR}$, gT_{LR}], [*goInto*$_{LR}$, $gIBE_{LR}$] [$gIBE_{LR}$, gT_{LR}], [$gIBE_{LR}$, $gIBE_{LR}$]
[(I-90, Urban), (I-90, Boston)]	**[*goInto*, *goInto*]**

7. Conclusions

This paper developed a series of qualitative spatial calculi based on the 9-intersection [5]. These calculi can be used for qualitative spatial reasoning on topological relations between various combinations of objects. Unlike many other calculi, the heterogeneous 9-intersection calculi are concerned with situations where multiple sorts of objects coexist in the same space. However, by integrating sets of base relations, composition tables, and converse lists, such heterogeneity is no longer an obstacle to conduct spatial reasoning in an algebraic framework.

In this work, we featured the 9-intersection because of its popularity in spatial database studies. However, the recent extension of the 9-intersection, called *the* 9^+-*intersection* [15, 19], serves as a more flexible framework for modeling topological relations. For instance, the 9^+-intersection can distinguish the topological relations between two directed lines in $\mathbb{R}^1$, which can be mapped to temporal relations between two intervals [15]. Thus, if we extend the 9^+-intersection into qualitative spatial calculi, the resulting 9^+-*intersection calculi* will cover temporal calculi as well (e.g., Allen's interval calculus [1]). In addition, the 9^+-intersection calculi will support topological relations between a directed line and a point/line/region in $\mathbb{R}^2$ and, accordingly, it will be useful for integrating knowledge about path-landmark arrangements collected by mobile agents. We are planning to develop such 9^+-intersection calculi and provide a library of both the 9- and 9^+-intersection calculi on SparQ [8] and CASL [20] for public use.

Another interesting future topic is to develop qualitative spatial calculi that feature non-topological relations (say, cardinal direction relations, relative orientations, distance relations, etc.) between multi-domain objects. We expect that we can use similar integration techniques for the development of such heterogeneous spatial calculi.

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