# From Three-Dimensional Topological Relations to Contact Relations

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## Abstract

Topological relations, which concern how two objects intersect, are one of the most fundamental and well-studied spatial relations. However, in reality, two physical objects can take only *disjoint* relation if they are solid. Thus, we propose an alternative of topological relations, called *contact relations*, which capture how two objects contact each other. Following the framework of the 9-intersection, this model distinguishes contact relations based on the presence or absence of contacts between several surface elements of two objects. Consequently, the contact relations have a strong correspondence to the 9-intersection-based topological relations. Making use of this correspondence, we derive the contact relations between various combinations of objects in a 3D Euclidean space ($\mathbb{R}^3$). For this purpose, we first review and analyze the topological relations in $\mathbb{R}^3$. Then, these topological relations are mapped to contact relations.

## 1 Introduction

Qualitative spatial relations are fundamentals for characterizing spatial arrangements of objects in a space, usually following how people conceptualize them. Spatial database communities have studied several types of qualitative spatial relations, among which topological relations have attracted much attention due to its importance in human spatial cognition (Shariff, et al. 1998). Egenhofer (1989), for instance, distinguished eight types of topological relations between two regions in a 2D Euclidean space $\mathbb{R}^2$, called *disjoint*, *meet*, *overlap*, *covers*, *coveredBy*, *contains*, *inside*, and *equal* relations. The same set of eight relations holds between two bodies in $\mathbb{R}^3$ (Zlatanova 2000). Similarly, Randell *et al*. (1992) distinguished eight types of topological relations without regard to the dimension of the underlying space. Such topological relations, however, fail when they are applied to physical objects—simply because they are usually solid and, accordingly, they do not intersect/connect with each other (note that the connection of two objects in Randell *et al*.'s sense is established by the presence of a common point, thereby different from the notion of contact).

Although the solid objects are always topologically disjoint (disconnected) in theory, people often care about whether they are totally separated or having a contact and, if they have a contact, in what way. For instance, we use the expressions like *the bottom of A meets the top of B* (or simply *A is on B*, although this expression has more broad meanings (Feist 2000)) or *A stands on the edge/corner of B* (Billen and Kurata 2008). These expressions concern which part of one object contacts which part of another object. Such information is critical when we characterize the spatial arrangement of two contacting objects. We, therefore, propose a new model of qualitative spatial relations that captures how two objects contact each other, called *contact relations*.

Contact may occur only between the surfaces of two objects. In a 3D space, for each region, we can distinguish its front and back sides. Similarly, we can distinguish several sides of bodies considering their shapes (cubes, pyramids, etc.). Thus, the model of contact relations should be able to record the presence or absence of contacts between several surface elements of two objects in different granularities. Our model, therefore, follows the frameworks of the *9-intersection* (Egenhofer and Herring 1991) and the *9⁺-intersection* (Kurata and Egenhofer 2007), which distinguish topological relations based on the presence or absence of intersections between the topological parts of two objects. Naturally, contact relations in our model have a strong correspondence with 9-intersection-based topological relations. Indeed, making use of this correspondence, we derive contact relations between various combinations of objects (Section 5).

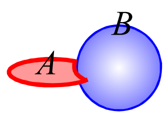
The remainder of this paper is organized as follows: Section 2 summarizes the concepts of the 9- and $9^+$-intersection. Based on these models, Section 3 develops a model of contact relations. Section 4 reviews topological relations in $\mathbb{R}^3$ and discusses the property of these relations for later discussions. Section 5 derives six sets of contact relations in $\mathbb{R}^3$ from the corresponding sets of topological relations discussed in Section 4. Finally, Section 6 concludes with a discussion of future problems.

In this paper, *solid objects* refer to the objects that do not intersect. We can consider that any sort of objects (points, lines, regions, and bodies) may be solid or non-solid objects. Our target is limited to solid objects. In addition, to simplify, we consider only simple objects (Schneider and Behr 2006). Simple objects consist of a single connected component. Simple lines are non-branching lines without loops, and simple regions/bodies are regions/bodies without holes, spikes, cuts, and disconnected interior.

## 2 The 9-Intrersection and The 9$^{+}$-Intersection

Our model of contact relations follows the 9-intersection (Egenhofer and Herring 1991) and its refinement called the 9$^{+}$-intersection (Kurata and Egenhofer 2007). Based on point-set topology (Alexandroff 1961), these two models distinguish the *interior*, *boundary*, and *exterior* each object. The interior of an object *X*, denoted $X°$, is the union of all open sets contained in *X*, *X*'s boundary $\partial X$ is the difference between *X*'s closure (the intersection of all closed point sets that contain *X*) and $X°$, and *X*'s exterior $X^{-}$ is the complement of *X*'s closure. By definition, the boundary of a line refers to the set of its two endpoints, and a point does not have an interior. In the 9-intersection, topological relations are characterized by the properties of the 3×3 intersections between the topological parts (interior, boundary, and exterior) of two objects. These 3×3 intersections are concisely represented in the *9-intersection matrix* in Eq. 1. In the most basic approach, topological relations are distinguished by the presence or absence of these 3×3 intersections. Accordingly, topological relations are characterized by the pattern of bitmap-like icons (Mark and Egenhofer 1994), whose 3×3 cells indicate the emptiness/non-emptiness of the 3×3 elements in the 9-intersection matrix (Fig. 1).

$$\mathrm{M}(A,B)=\begin{pmatrix} A°\cap B° & A°\cap\partial B & A°\cap B^{-} \\ \partial A\cap B° & \partial A\cap\partial B & \partial A\cap B^{-} \\ A^{-}\cap B° & A^{-}\cap\partial B & A^{-}\cap B^{-} \end{pmatrix} \qquad \textbf{(1)}$$




$$\mathrm{M}(A,B)=\begin{pmatrix} \emptyset & \emptyset & \neg\emptyset \\ \emptyset & \neg\emptyset & \neg\emptyset \\ \neg\emptyset & \neg\emptyset & \neg\emptyset \end{pmatrix} \rightarrow$$

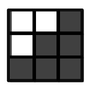
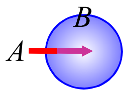


**Fig. 1.** Representations of a topological relation by the 9-intersection matrix/icon

The 9$^{+}$-intersection extends the 9-intersection by considering the *subdivision* of topological parts. For instance, the boundary of a line can be subdivided into two endpoints. To support such subdivision, the nested version of the 9-intersection matrix shown in Eq. 2, called the *9⁺-intersection matrix*, is used, where $A^{\circ i}$, $\partial_i$A, and $A^{-i}$ are the $i^{\text{th}}$ subpart of $A$'s interior, boundary, and exterior, while $B^{\circ j}$, $\partial_j$B, and $B^{-j}$ are the $j^{\text{th}}$ subpart of $B$'s interior, boundary, and exterior, respectively. Just like the 9-intersection, topological relations are distinguished by the presence or absence of all intersections listed in the $9^+$-intersection matrix. For instance, Fig. 2 shows the $9^+$-intersection matrix and its iconic representation of a topological relation between a (directed) line and a region. To simplify, the bracket of each matrix element is omitted if it consists of a single sub-element.

$$\mathrm{M}^+(A,B) = \begin{pmatrix} [A^{\circ i} \cap B^{\circ j}] & [A^{\circ i} \cap \partial_j \mathrm{B}] & [A^{\circ i} \cap B^{-j}] \\ [\partial_i \mathrm{A} \cap B^{\circ j}] & [\partial_i \mathrm{A} \cap \partial_j \mathrm{B}] & [\partial_i \mathrm{A} \cap B^{-j}] \\ [A^{-i} \cap B^{\circ j}] & [A^{-i} \cap \partial_j \mathrm{B}] & [A^{-i} \cap B^{-j}] \end{pmatrix} \qquad \mathbf{(2)}$$




$$\mathrm{M}^+(A,B) = \begin{pmatrix} A^\circ \cap B^\circ & A^\circ \cap \partial B & A^\circ \cap B^- \\ \begin{bmatrix} \partial_1 A \cap B^\circ \\ \partial_2 A \cap B^\circ \end{bmatrix} & \begin{bmatrix} \partial_1 A \cap \partial B \\ \partial_2 A \cap \partial B \end{bmatrix} & \begin{bmatrix} \partial_1 A \cap B^- \\ \partial_2 A \cap B^- \end{bmatrix} \\ A^- \cap B^\circ & A^- \cap \partial B & A^- \cap B^- \end{pmatrix} = \begin{pmatrix} \neg\emptyset & \neg\emptyset & \neg\emptyset \\ \begin{bmatrix} \emptyset \\ \neg\emptyset \end{bmatrix} & \begin{bmatrix} \emptyset \\ \emptyset \end{bmatrix} & \begin{bmatrix} \neg\emptyset \\ \emptyset \end{bmatrix} \\ \neg\emptyset & \neg\emptyset & \neg\emptyset \end{pmatrix} \rightarrow$$

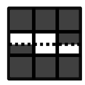


**Fig. 2.** Representations of a topological relation by the $9^+$-intersection matrix/icon

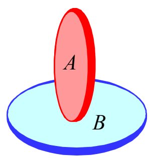
## 3 Modeling Contact Relations

Contact relations between two objects are distinguished basically based on the presence or absence of contacts between their surface elements. Surface elements of an object are the parts of the object that is connected directly to the object's exterior. For regions in $\mathbb{R}^3$, we can naturally distinguish three surface elements; the front side, the back side, and the edge. The front and back sides correspond to the region's topological interior, while the edge corresponds to the region's topological boundary. For bodies, we may distinguish different numbers of surface elements; for instance, we can distinguish six sides of a cubic body. To simplify, however, this paper does not distinguish the sides of bodies and considers that the body has only one surface element, which corresponds to the body's topological boundary. For lines, we consider that they have two surface elements, a set of two endpoints and an intermediate segment, which correspond to the line's topological boundary and interior, respectively.

In this way, surface elements of objects are distinguished in association with the topological part of the objects. Making use of this association, we use a $9^+$-intersection-like matrix, called the *$9^+$-contact matrix* (Eq. 3), for representing contact relations. In this matrix, each Boolean function $\mathrm{c}(X, Y)$ indicates whether $X$ contacts $Y$ or not. $X$ and $Y$ may refer to a surface element of an object, which we define as a subpart of the object's interior or boundary. $X$ and $Y$ may also refer to the exterior of an object. By recording the presence or absence of contact between $A$'s surface elements (say $X$) and $B$'s exterior $B^-$, we can tell from the matrix whether $X$ contacts the surface of another object entirely, partly, or not at all—for instance, if $X$ contacts only one of $B$'s surface element $Y$ and not $B^-$, $X$ entirely contacts $Y$. The most right-bottom element in the $9^+$-contact matrix is fixed, as it does not contribute to the distinction of contact relations. This does not necessarily mean that we should interpret that the exteriors of two objects contact each other.

$$\mathrm{C}^+(A,B) = \begin{pmatrix} [\mathrm{c}(A^{\circ_i}, B^{\circ_j})] & [\mathrm{c}(A^{\circ_i}, \partial_j B)] & [\mathrm{c}(A^{\circ_i}, B^-)] \\ [\mathrm{c}(\partial_i A, B^{\circ_j})] & [\mathrm{c}(\partial_i A, \partial_j B)] & [\mathrm{c}(\partial_i A, B^-)] \\ [\mathrm{c}(A^-, B^{\circ_j})] & [\mathrm{c}(A^-, \partial_j B)] & true \end{pmatrix} \quad (3)$$

Since all elements in the $9^+$-contact matrix are two-valued (*true* or *false*), the patterns of the $9^+$-contact matrix are represented by bitmap-like icons (Fig. 3) in which each black/white cell indicates the true/false value of the corresponding element in the $9^+$-contact matrix. In the example of Fig. 3 the icon's first column and first row are partitioned, since they correspond to the front and back sides of the regions.




$$\mathrm{C}^+(A,B) = \begin{pmatrix} \begin{bmatrix} false & false \\ false & false \end{bmatrix} & \begin{bmatrix} false \\ false \end{bmatrix} & \begin{bmatrix} true \\ true \end{bmatrix} \\ [true \;\; false] & false & true \\ [true \;\; true] & true & true \end{pmatrix} \rightarrow$$

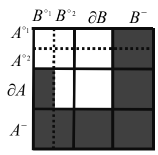



**Fig. 3.** Representations of a contact relation by the $9^+$-contact matrix/icon

An attention should be paid to the definition of the contact between the boundary (edge) of a region and the exterior of another object. We say that the boundary of a region $A$ contacts the exterior of an object $B$ if and only if $A$'s boundary has an interval that does not contact $B$'s surface. Accordingly, in Fig. 4, we consider that $A$'s edge does not contact $B$'s exterior.

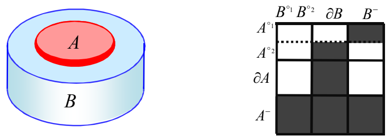



**Fig. 4.** An example case where we consider that a region's boundary does not contact a body's exterior

Dimensional model by Billen *et al.* (2002) also considers the distinction of surface elements (corner points, edges, and faces) for distinguishing *dimensional relations*, which are essentially a refinement of topological relations. Later Billen and Kurata (2008) reformulated the dimensional model using the framework of the $9^+$-intersection, giving a new name *projective $9^+$-intersection*. Our model is structurally similar to this projective $9^+$-intersection, although our model targets the contacts while the projective $9^+$-intersection deals with the intersections. In addition, the front and back sides of regions are not distinguished in the projective $9^+$-intersection.

## 4 Three-Dimensional Topological Relations

This section reviews and analyzes topological relations in $\mathbb{R}^3$, which are later used for deriving contact relations. The topological relations in $\mathbb{R}^3$ were studied by Zlatanova (2000). Later, Kurata (2008) systematically re-identified the sets of topological relations between various combinations of objects in $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$, making use of the universal constraints on the $9^+$-intersection. Table 1 shows the numbers of the identified relations. Among these relation sets, this paper uses point-body relations (Fig. 5), line-region relations (Fig. 6), line-body relations (Fig. 7), region-region relations (Fig. 8), region-body relations (Fig. 7), and body-body relations (Fig. 9), each of which contains the relations that hold only in $\mathbb{R}^3$.

**Table 1.** Numbers of topological relations between points, simple lines, simple regions, and simple bodies in $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$, distinguished by the emptiness/non-emptiness patterns of the 9-intersection (Kurata 2008)

| | $\mathbb{R}^1$ | $\mathbb{R}^2$ | $\mathbb{R}^3$ |
|---|---|---|---|
| Point-Point | 2 | 2 | 2 |
| Point-Line | 3 | 3 | 3 |
| Point-Region | – | 3 | 3 |
| Point-Body | – | – | 3 |
| Line-Line | 8 | 33 | 33 |

| | $\mathbb{R}^1$ | $\mathbb{R}^2$ | $\mathbb{R}^3$ |
|---|---|---|---|
| Line-Region | – | 19 | 31 |
| Line-Body | – | – | 19 |
| Region-Region | – | 8 | 44 |
| Region-Body | – | – | 19 |
| Body-Body | – | – | 8 |

### 4.1 Correspondence between Topological Relation Sets

Comparison of the relation sets in Figs. 5-9 reveals several correspondences between the relation sets. The most remarkable correspondence is that between the 19 line-body relations and the 19 region-body relations, which are represented by almost the same sets of icons (Fig. 7). The 19 line-body relations also correspond to the 19 of the 31 line-region relations in Figs. 6a-s. Note that these 19 line-region relations are the relations that may hold also in $\mathbb{R}^2$. Considering that both the boundary of a body and that of a region forms a Jordan curve in $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively, the correspondence between the line-body relations in $\mathbb{R}^3$ and the line-region relations in $\mathbb{R}^2$ is reasonable. For the same reason, the 8 body-body relations (Fig. 9) correspond to the 8 of the 43 region-region relations in Figs. 8a-$f_{1/2}$, which may hold also in $\mathbb{R}^2$.

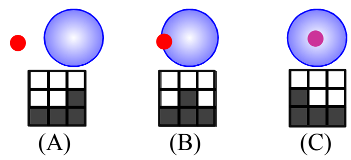



**Fig. 5.** 3 topological point-body relations in $\mathbb{R}^3$

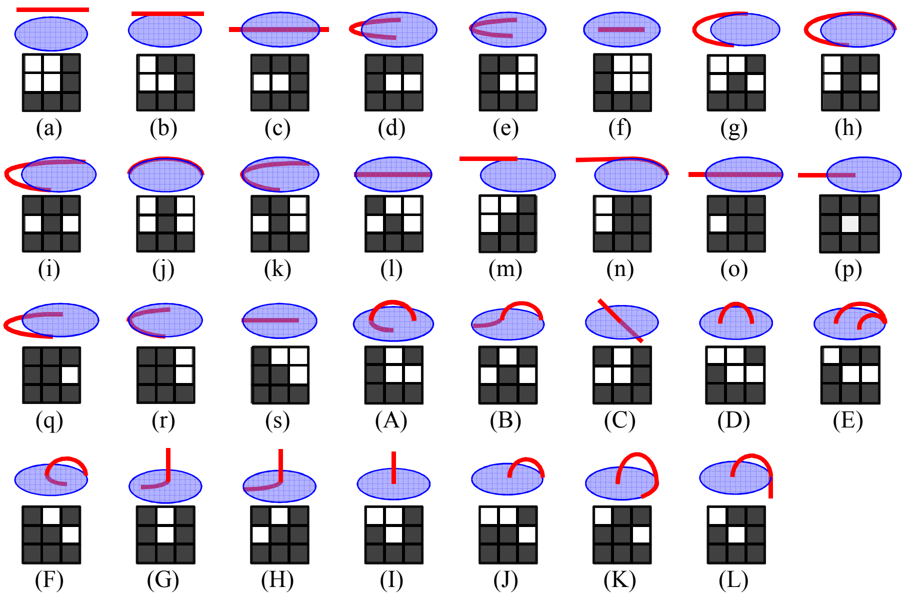



**Fig. 6.** 31 topological line-region relations in $\mathbb{R}^3$

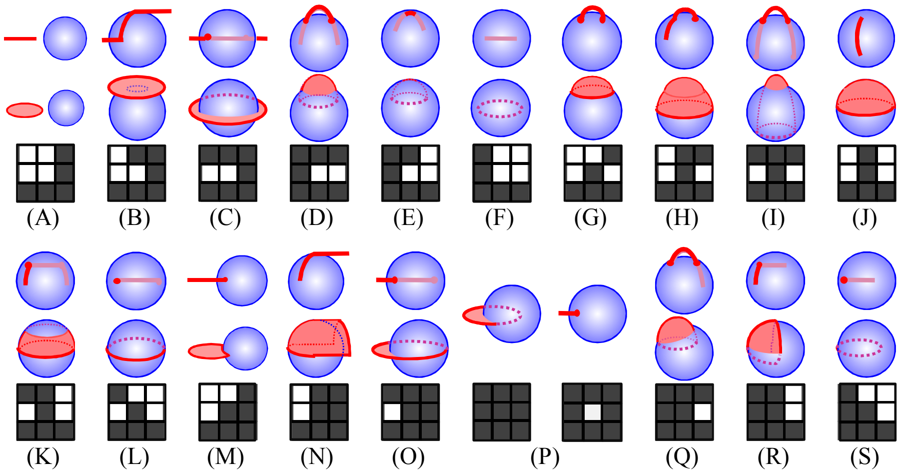



**Fig. 7.** 19 topological line-body and region-body relations in $\mathbb{R}^3$

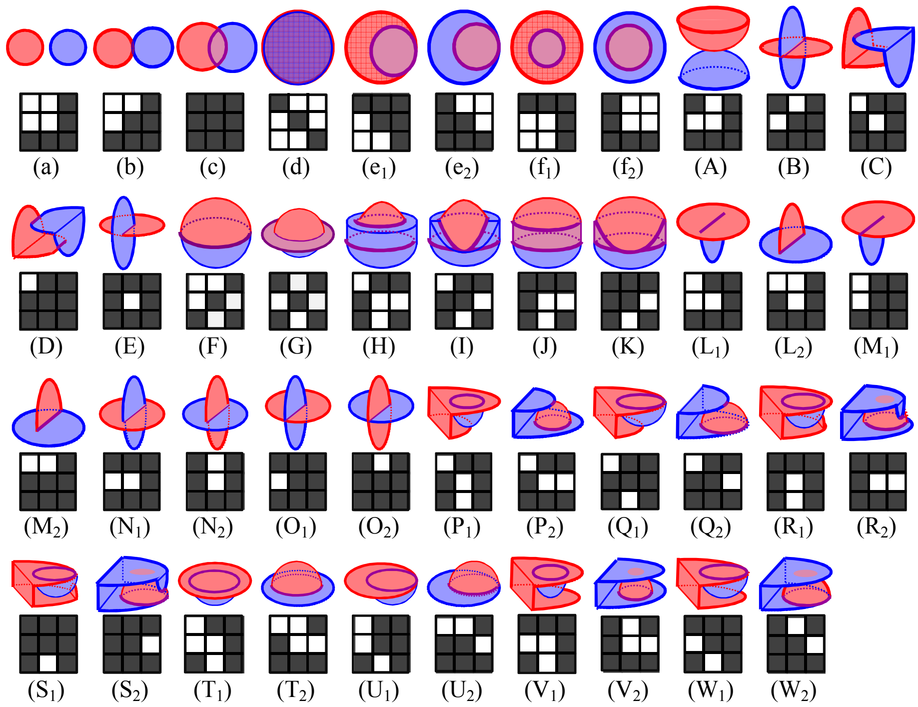



**Fig. 8.** 43 topological region-region relations in $\mathbb{R}^3$

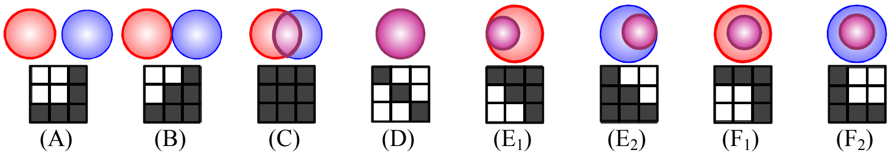



**Fig. 9.** 8 topological body-body relations in $\mathbb{R}^3$

## 4.2 Topological Relations That Hold between Convex Objects

For each topological relation set, we identify the relations that hold even when the objects are limited to convex ones. For instance, the line-region relations in Figs. 8a-$f_{1/2}$ hold between two convex regions, while the relation in Fig. 8A does not. The reason why we pay attention to such relations between convex objects, since convex objects are *prototypical*—convex lines are straight and convex regions are flat—and also many physical objects in the real world are convex.

The topological relations that cannot hold between two convex objects (say, *A* and *B*) are filtered out by the following constraints:

- if *A*'s boundary intersects with *B*'s interior only or with *B*'s interior and boundary, but not *B*'s exterior, then *A*'s interior intersects with *B*'s interior only (and vice versa);
- if *A*'s boundary intersects with *B*'s boundary only, then *A*'s interior intersects with *B*'s interior only or *B*'s boundary only (and vice versa);
- *A*'s interior and *B*'s interior intersect if and only if *A*'s boundary intersects with *B*'s closure or *B*'s boundary intersects with *A*'s closure.

By sketching an instance manually it is confirmed that the remaining topological relations hold between two convex objects. In this way, we identified the following topological relations that hold even when the objects are limited to convex ones:

- all point-body relation (Figs. 5A-C);
- 13 line-region relations in Figs. 6a-c,f,j,l-p,s,C,I;
- 11 line-body relations in Figs. 7A-C,F,J,L-P,S;
- 11 region-body relations in Figs. 7A-C,F,J,L-P,S;
- 18 region-region relations in Figs. 8a-$f_{1/2}$,,B,$L_{1/2}$-$O_{1/2}$; and
- all body-body relations (Figs. 9A-$F_{1/2}$).

## 4.3 Topological Relations with Piercing

We say that an object *A* pierces an object *B* when *A*'s interior intersects continuously with *B*'s interior and its neighboring exterior on two different sides. In our framework, the pierced object *B* is limited to a region, since the sides are defined only for regions (Section 3). We pay attention to such piercing because topological relations that require piercing cannot be mapped to contact relations (Sections 5.3-5.4). Piercing are found in the sample configurations of the relations in Fig. 6C, Figs. 7C-F,I,K,L,O-S, and Figs. 8B,E,$N_{1/2}$-$O_{1/2}$. This does not mean that these relations require piecing. In fact, Fig. 10 shows that the line-region relation in Fig. 6C and the region-region relations in Figs. 8B,E,$N_{1/2}$-$O_{1/2}$ do not require piercing (when two objects are limited to convex ones, however, these topological relations require piercing). On the other hand, the region-body relations in Figs. 7C-F,I,K,L,O-S always require piercing, because whenever the body's interior intersects with the region's interior, the body's interior also intersects continuously with the region's neighboring exterior on both sides.

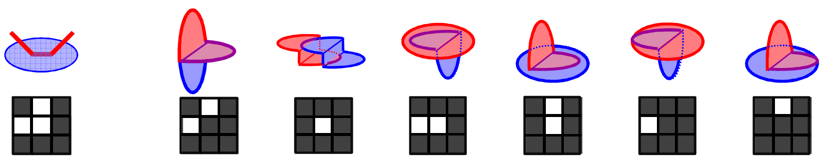



**Fig. 10.** Alternative configurations of the line-region relation in Fig. 6C and the region-region relations in Figs. 8B,E,$O_{1/2}$-$M_{1/2}$, which do not have piercing.

### 4.4 Topological Relations with Enclosed Spaces

In Figs. 8F-K,$P_{1/2}$-$W_{1/2}$, we can find a space (or spaces) enclosed by two regions. Actually, any configuration of the region-region relations in Figs. 8F-K,$P_{1/2}$-$W_{1/2}$ have enclosed space(s), because the closure of one region contains the boundary of another region, but not its interior. Similarly, any configuration of the region-body relations in Figs. 7D,G,H,Q have enclosed space(s). The presence of such enclosed space imposes some restrictions on the corresponding contact relations (Section 5.4).

## 5 Deriving Contact Relations

Contact relations have a strong correspondence with topological relations, as we can imagine from the structural similarity between the $9^+$-contact matrix and the 9-/$9^+$-intersection matrix. Normally, contact relations can be derived from topological relations simply by replacing intersections with contacts (Figs. 11a-b). As for topological relations between a region and an object $X$, each topological relation is potentially mapped to multiple contact relations, since the intersection between the region's interior and $X$'s interior/boundary is mapped to contacts between either or both faces of the region and $X$'s interior/boundary (Fig. 11b). On the other hand, there are some topological relations that cannot be mapped to contact relations , such as those with piercing (Fig. 11c). In the following subsections, we derive the sets of contact relations between various combinations of objects based on the mapping from the corresponding sets of topological relations.

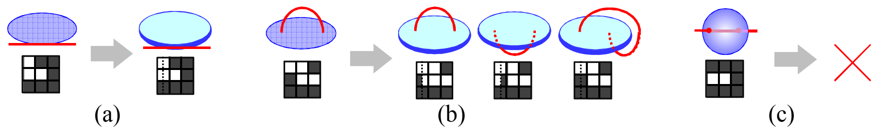



**Fig. 11.** Sample mappings from topological relations to contact relations

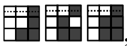
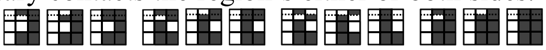
### 5.1 Point-Body, Line-Body, and Body-Body Relations

The topological point-body relations in Fig. 5A,B are mapped to one contact relation, respectively, while the relation in Fig. 5C cannot be mapped to contact relations because the point is inside the body. Consequently, two contact relations between a point and a body in $\mathbb{R}^3$, represented by the $9^+$-contact icons , are derived. In a similar way, seven contact relations between a line and body in $\mathbb{R}^3$, , are derived from the topological line-body relations in Figs. 7A,B,G,H,J,M,N, and two contact relations between two bodies in $\mathbb{R}^3$, , are derived from the topological body-body relations in Figs. 9A,B.

Among these contact relations, which hold when the objects are limited to convex ones? Section 4.2 found that the topological point-body relations in Fig. A,B, the topological line-body relations in Figs. 7A,B,J,M,N, and the topological body-body relations in Figs. 9A,B hold in convex cases. Naturally, the contact relations derived from these topological relations are the relations that hold in convex cases; i.e.,

- two contact relations, , hold between a point and a convex body;
- five contact relations, , hold between a convex line and convex body; and
- two contact relations, , hold between two convex bodies.

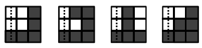
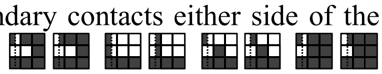
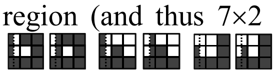
### 5.2 Region-Body Relations

Contact relations between a region and a body in $\mathbb{R}^3$ are derived from the 19 topological region-body relations in Fig. 7. The relations in Figs. 7A,G,M are mapped to one contact relation, respectively. Accordingly, 3×1 contact relations, , are derived from them. The relations in Figs. 7B,H,J,N are mapped to three contact relations, respectively, because the body's boundary contacts the region's either or both sides. Thus, 4×3 contact relations, , are derived from them. Finally, the remaining relations, where the body's interior intersects with the region's closure, cannot be mapped to contact relations. In this way, in total 3×1+4×3 = 15 contact relations between a region and a body in $\mathbb{R}^3$ are derived.

Among these 15 contact relations, the relations that hold in convex cases are limited to those derived from the topological region-body relations that hold in convex cases—i.e., the relations in Figs. 7A,B,J,M,N (Section 4.2). At this time the relations in Figs. 7B,J,N are mapped to not three, but two contact relations, respectively, because the boundary of a convex body may contact either side of a convex region, but not both sides. Consequently, 2×1+3×2 = 8 contact relations, , hold between a convex region and a convex body in $\mathbb{R}^3$.

### 5.3 Line-Region Relations

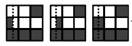
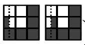
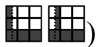
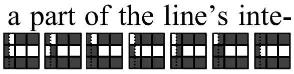
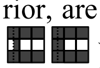
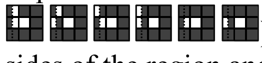
The mapping from topological line-region relations to contact relations is a bit complicated, because the line's interior and boundary may contact the region's either or both sides, in some cases not independently. To simplify, we start from the relations between a convex line and a convex region. Section 4.2 found that there are 13 topological line-region relations that hold in convex cases (Figs. 6a-c,f,j,l-p,s,C,I). Among these relations,

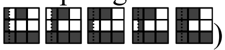
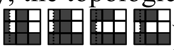
- the five relations in Figs. 6a,b,j,m,n, where the region's interior does not intersect with the line's interior or boundary, are mapped to one contact relation, respectively (and thus 5×1 contact relations   are derived);
- the seven relations in Figs. 6c,f,l,o,p,s,I, where the region's interior intersects with the line's interior/boundary, are mapped to two contact relations in which the line's interior/boundary contacts either side of the region (and thus 7×2 contact relations   are derived); and

- the relation in Fig. 6C cannot be mapped to a contact relation because it requires piercing.

In this way, 5×1+7×2 = 19 contact relations between a convex line and a convex body in $\mathbb{R}^3$ are derived.

If the line and the body are not limited to convex ones, contact relations are derived as follows:

- the seven topological relations in Figs. 6a,b,g,h,j,m,n, where the region's interior does not intersect with the line's interior or boundary, are mapped to one contact relation (e.g., ), respectively;
- the seven topological relations in Figs. 6c,i,k,o,B,C,H, where the region's interior intersects with a part of the line's interior and not the line's boundary, are mapped to three contact relations (e.g., ) in which the line's interior contacts the region's either or both sides, respectively;
- the two topological relations in Figs. 6D,E, where the region's interior intersects with both of the line's endpoints and not the line's interior, are mapped to three contact relations (e.g., ) in which the line's two endpoints contact the region's either or both sides, respectively;
- the four topological relations in Figs. 6I-L, where the region's interior intersects with one of the line's endpoints and not the line's interior, are mapped to two contact relations (e.g., ) in which the line's one endpoint contacts the region's either side, respectively;
- the three topological relations in Figs. 6f,l,s, where the region's interior contains the line's interior, are mapped to two contact relations (e.g., ) in which the line's interior contacts either side of the region and the line's endpoint(s) follows it, respectively;
- the two topological relations in Figs. 6d,A, where the region's interior intersects with both of the line's endpoints and a part of the line's interior, are mapped to 3×3 contact relations (e.g.,   ) in which the line's interior and boundary contact either or both sides of the region independently, respectively;
- the four topological relations in Figs. 6p,q,F,G, where the region's interior intersects with one of the line's endpoints and a part of the line's interior, are mapped to 3×2 contact relations (e.g., ) in which the line's interior contacts either or both sides of the region and the line's one endpoint contacts either side of the region independently, respectively;

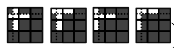
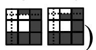
- the topological relation in Fig. 6e is mapped to five contact relations () where the line's interior contact either or both sides of the region and the line's endpoints follow it; and
- similarly, the topological relation in Fig. 6r is mapped to four contact relations ().

In this way, in total 7×1+7×3+4×3+2×2+3×2+2×9+4×6+5+4 = 99 contact relations between a line and a region in $\mathbb{R}^3$ are derived

### 5.4 Region-Region Relations

The mapping from topological region-region relations to contact relations is more complicated than the previous case, because two sides and edge of one region may contact either or both sides of another region, in many cases not independently. The dependency occurs in the mapping from the topological relations in Figs. 8F-K,$P_{1/2}$-$W_{1/2}$ and those in Figs. 8d-$f_{1/2}$. In the former relations, the configurations always have enclosed space(s) (Section 4.4). Consequently, the side of one object that entirely faces the enclosed space cannot contact the side of another object that does not face the enclosed space. In the latter relations, one region $X$ is contained by another region $Y$. Consequently, in their corresponding contact relations, if $X$'s either or both sides entirely contacts $Y$'s $\alpha$-side, then $X$'s boundary also contacts $Y$'s $\alpha$-side. Considering these constraints, we derived 1051 contact relations between two regions in $\mathbb{R}^3$ from the 43 topological region-region relations in $\mathbb{R}^3$ (See Appendix for the detail).

When two regions are limited to convex ones, the mapping becomes much simpler. Section 4.2 found 18 topological region-region relations that hold between two convex regions in $\mathbb{R}^3$ (Figs. 8a-$f_{1/2}$, B, E, and $L_{1/2}$-$O_{1/2}$). These relations are mapped to contact relations as follows:

- The two relations in Figs. 8a,b, which have no interior-related intersection, are mapped to one contact relation (e.g., ), respectively;
- The six relations in Figs. 8c-$f_{1/2}$, which have an interior-interior intersection, are mapped to 2×2 contact relations (e.g., ) in which either side of one region contacts either side of another region, respectively;
- The four relations in Figs. $8L_{1/2}$-$M_{1/2}$, which have a boundary-interior intersection but no interior-interior intersection, are mapped to two contact relation (e.g., ) in which the boundary of one region contacts either side of another region, respectively;

- The six relations in Figs. 8B, E, and $N_{1/2}$-$O_{1/2}$ cannot be mapped to contact relation because they require piercing (Section 4.3).

Consequently, in total 2×1+6×4+4×2 = 34 contact relations between two convex regions in $\mathbb{R}^3$ are derived. These 34 contact relations are illustrated schematically in Fig. 12. This figure is actually a *conceptual neighborhood graph* (Freksa 1992) of the 34 contact relations, whose links show pairs of *similar* relations (in the sense that the configuration of one relation may switch to another relation by a minimal change). We can see nice horizontal and vertical symmetries in this graph, which result from the fact that each region has two sides.

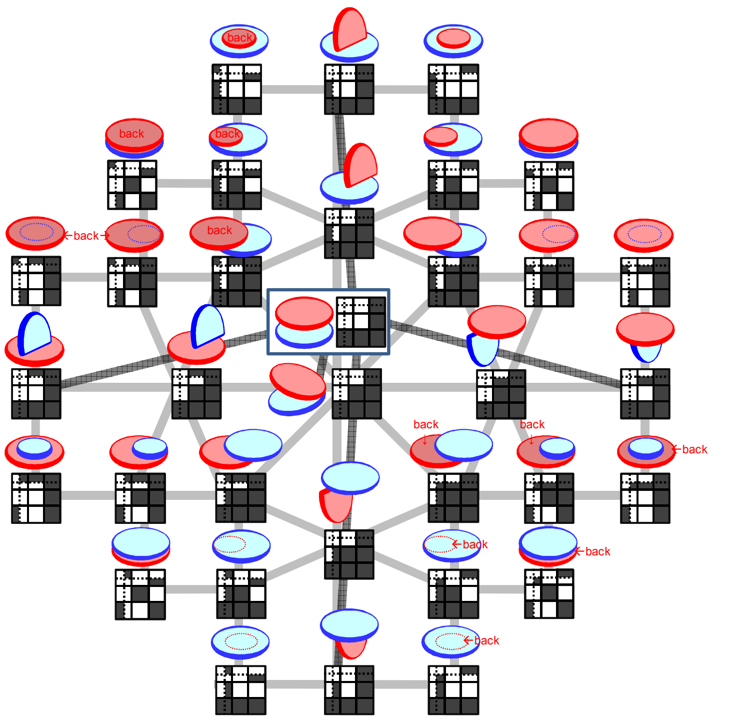



**Fig. 12.** A conceptual neighborhood graph of the 34 contact relations between two convex regions in $\mathbb{R}^3$

## 6 Conclusions and Future Work

This paper proposed a model of contact relations. The contact relations work as an alternative of topological relations when the target objects are solid. Due to the models' schematic similarity, contact relations have a strong correspondence with topological relations distinguished by the 9-intersection. Making use this correspondence, the sets of contact relations between various combinations of objects in $\mathbb{R}^3$ are derived from the corresponding sets of topological relations. The numbers of the derived relations (2 for point-body relations, 7 for line-body relations, 2 for body-body relations 15 for region-body relations, and 99 for line-region relations) are probably cognitively and practically acceptable, but unfortunately that for region-region relations (1051) is overwhelming. In general, if the objects are limited to convex ones, the numbers of the contact relations become much smaller (2 for point-body relations, 5 for line-body relations, 2 for body-body relations, 8 for region-body relations, 19 for line-region relations, and 34 region-region relations). One remaining issue is to analyze the completeness of the derived relations (i.e., to check the absence of additional relations).

This paper assumed that each body has only one surface element, but it is often possible to distinguish multiple surface elements of the body considering its shape (a cube, a pyramid, etc.). The $9^+$-contact matrix allows the distinction of surface elements in arbitrary granularity, thanks to its nested structure. Of course, it is expected that the distinction of more surface elements yields rapid increase of the number of contact relations, but the model itself is useful for describing how objects contact.

One interesting future topic is spatial reasoning based on contact relations. It is expected that the composition rules of contact relations are considerably different from those of topological relations. For instance, the composition rule of topological relations "*if X includes whole Y and Y includes whole Z, then X includes whole Z*" (Egenhofer 1994) cannot be transformed to the composition rule of contact relations, because if *X* contacts whole *Y*, then *Y* cannot contact *Z*.

Another interesting future topic is to identify and analyze contact relations in $\mathbb{R}^2$. In a 2D space we can distinguish *left* and *right* sides of lines. 2D contact relations can be used for characterizing the spatial arrangements of two streets and, accordingly, reasoning techniques on such 2D contact relations will be useful for mobile robots to develop the knowledge of street networks efficiently without exploring all streets.

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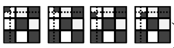
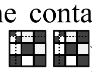
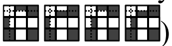
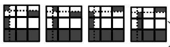
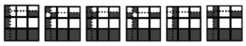
## Appendix: Mapping from Topological Region-Region Relations in $\mathbb{R}^3$ to Contact Relations

- the topological relations in Figs. 8a,b,F are mapped to only one contact relation, respectively, due to the lack of interior-related intersections;
- the topological relations in Fig. 8c, Figs. 8A-B, Figs. 8C-D, Fig. 8E, Figs. $8L_{1/2}$-$M_{1/2}$, and Figs. $8N_{1/2}$-$O_{1/2}$ are mapped to 15×3×3, 15, 3×3, 15×3×3, 3, and 15×3 contact relations, respectively, because in these mappings we can consider edge-side contacts and side-side contacts independently; that is,
  - if the topological relations have a boundary-interior intersection, they are mapped to $2^2-1 = 3$ types of edge-side contacts (i.e., the boundary of one region contacts either or both sides of another region); and
  - if the topological relations have an interior-interior intersection, this intersection is mapped to $2^4-1 = 15$ types of side-side contacts (i.e., either or both sides one region contact either or both sides of another region);
- the topological relations in Fig. 8G is mapped to six contact relations—four are the contact relations where *A*'s one side contacts *B*'s one side (i.e., ) and two are the contact relations where *A*'s two sides contact *B*'s different sides (i.e., );

- the topological relation in Fig. 8H is mapped to four contact relations in which the boundary of each region contacts either side of another region (i.e., );
- the topological relation in Fig. 8I is mapped to one more contact relation than the previous case, in which the boundary of each region contacts both sides of another region (i.e.,  in addition to );
- the topological relation in Fig. 8J is mapped to 24 contact relations, because once we assume that *A*'s *x*-side contacts *B*'s boundary and *B*'s *y*-side contacts *A*'s boundary (four possibilities exist), there are six possible side-side contacts (since *A*'s *x*-side and *B*'s *y*-side may contact either or both sides of another region, *A*'s non-*x*-side may contact *B*'s *y*-side, and *B*'s non-*y*-side may contact *A*'s *x*-side);
- the topological relation in Fig. 8K is mapped to 15 more contact relations than the previous case, in which the boundary of both regions contact the both sides of another region and both sides of each region may contact either or both sides of another region;
- the topological relations in Figs. $8T_{1/2}$ are mapped to two contact relations, in which the entire boundary of one region contacts either side of another region;
- the topological relations in Figs. $8U_{1/2}$ are mapped to one more contact relation than the previous case, in which one region *clips* another region (Fig. 13a) and, accordingly, the boundary of the clipping region contacts both sides of the clipped region;
- the topological relations in Figs. $8P_{1/2}$ are mapped to four contact relations, in which the boundary of each region contacts either side of another region;
- the topological relations in Figs. $8Q_{1/2}$ are mapped to three more contact relations than the previous case, in which one region clips another region and the boundary of the clipped region contacts either or both sides of another region;
- the topological relations in Figs. $8R_1,V_1$ are mapped to 28 contact relations, because once we assume that *A*'s *x*-side and *B*'s *y*-side encloses a space (four possibilities exist), there are seven possible side-side contacts (because *A*'s *x*-side may contact *B*'s side, while A's non-*x*-side may contact either or both sides of *B*), and similarly the topological relations in Figs. $8R_2,V_2$ are mapped to 28 contact relations;
- the topological relations in Figs. $8W_{1/2}$ are mapped to 15 more relations than those in Figs. $8V_{1/2}$ are, in which either or both sides of the clipped region contact either or both sides of another regions;

- the topological relations in Figs. $8S_{1/2}$ are mapped to 45 more relations than those in Figs. $8R_{1/2}$ are, in which the boundary and either or both sides of the clipped region contact either or both sides of another region independently;
- the topological relation in Fig. 8d is mapped to four contact relations in which either side of one region contacts either side of another region;
- the topological relation in Fig. $8f_1$ is mapped to 16 contact relations, among which four are the relations in which one entire side of *A* contacts either side of *B* (i.e., ) and twelve are the relation where one side of *A* entirely contacts either side of *B* and *A*'s another side partly contacts either or both sides of *B* (i.e.,  ), and similarly the topological relation in Fig. $8f_2$ is mapped to 16 contact relations; and
- the topological relation in Fig. $8e_1$ is mapped to eight more contact relations than that in Fig. $8f_1$, in which *A* is clipped by *B* plainly such that *B*'s one side contacts both sides of *A* (Fig. 13b) and *B*'s another side may contact either or both sides of *A*, and similarly, the topological relation in Fig. $8e_2$ is mapped to eight more contact relations than that in Fig. $8f_2$.

(a) (b)

**Fig. 1.** Configurations of two contact relations, derived from the topological region-region relations in (a) Fig. $8U_1$ and (b) Fig. $8e_1$, in which the region *A* is clipped by the region *B*