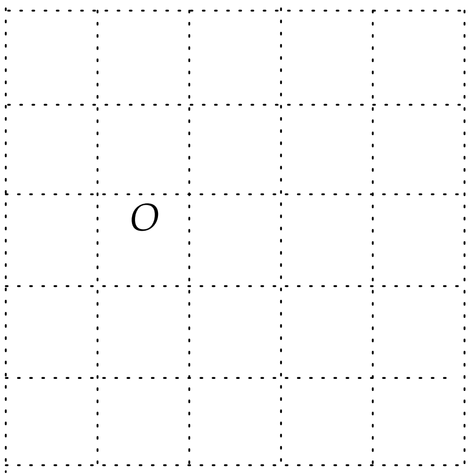
Loop-erased random walk on a fractal – a random fractal approach

Kumiko Hattori (Tokyo Metropolitan University), joint work with Michiaki Mizuno

0. What I’m going to talk about

Simple random walk on a graph

Jumps to a nearest neighbor with equal probability.



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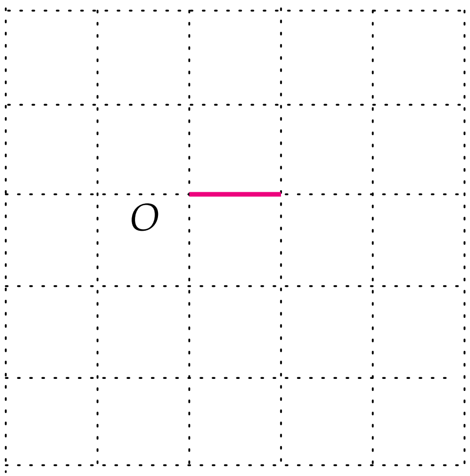

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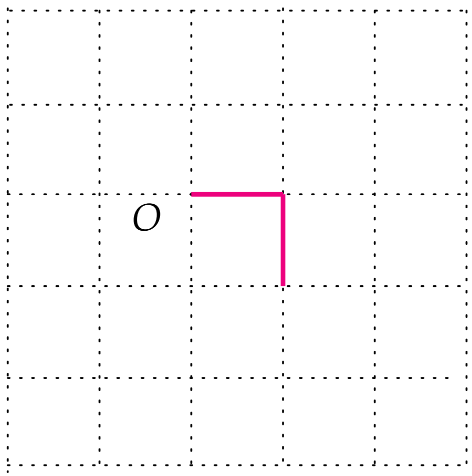

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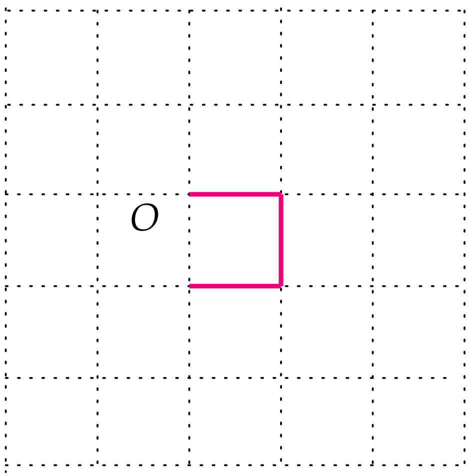

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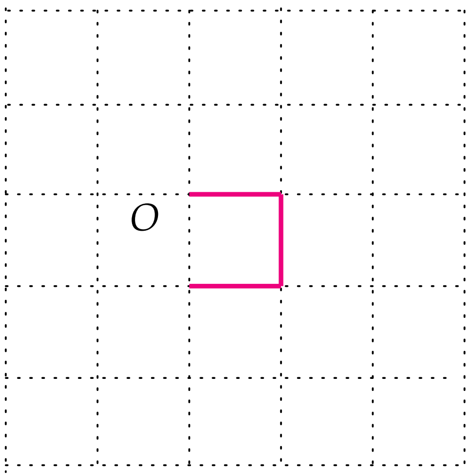

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Erase loops in chronological order.

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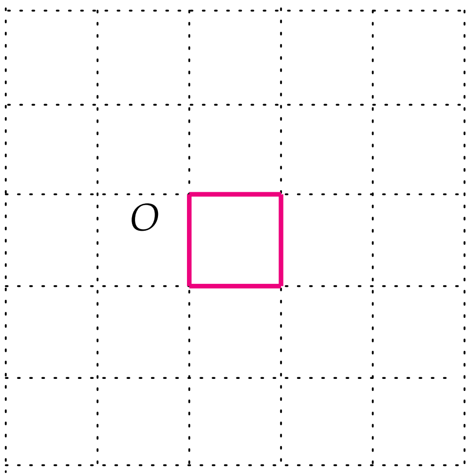

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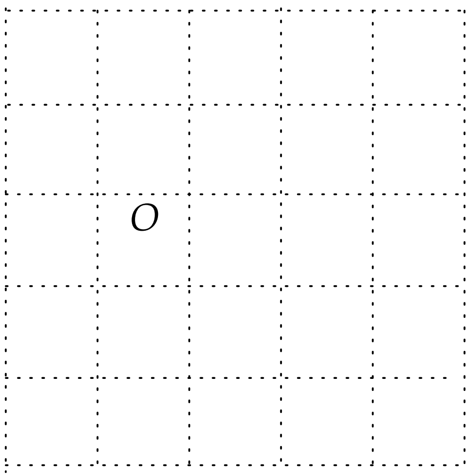

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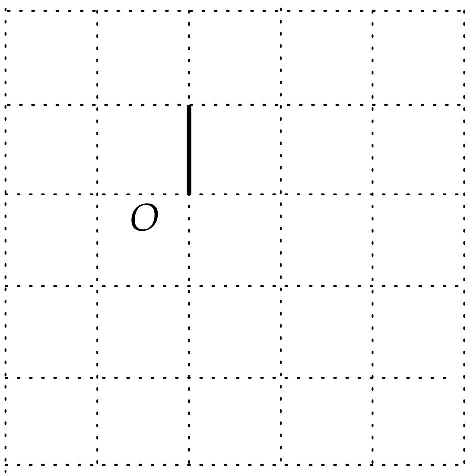

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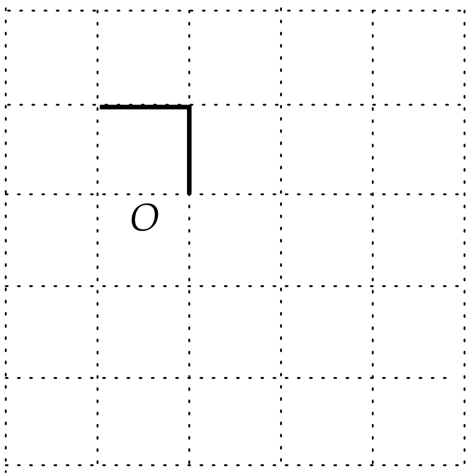

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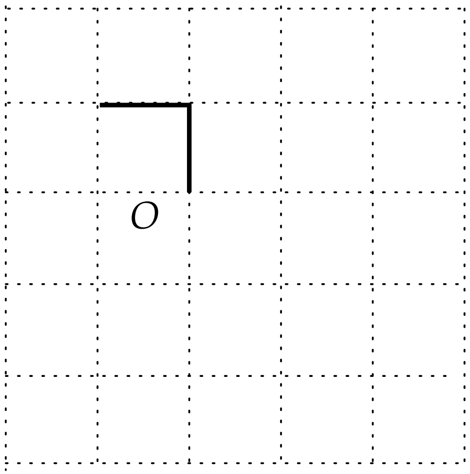

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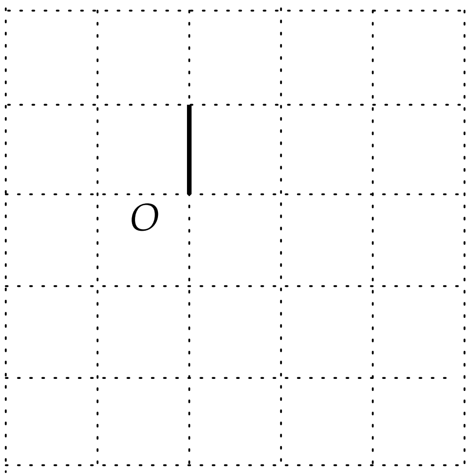

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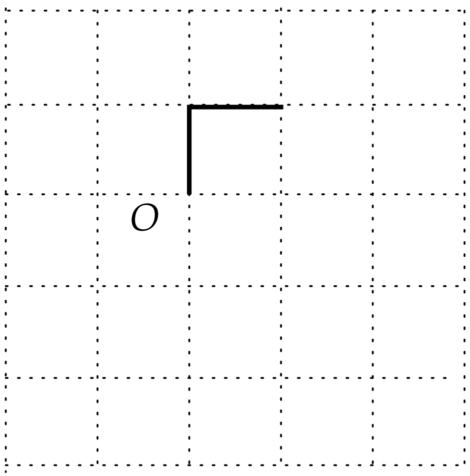

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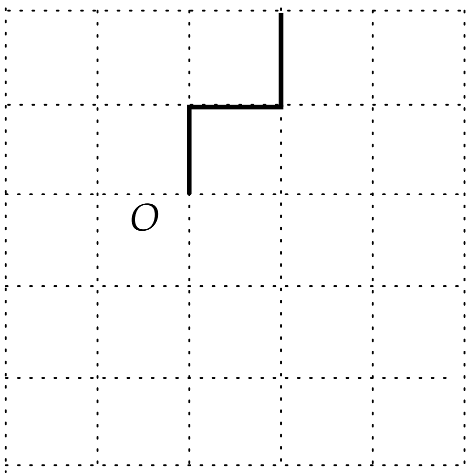

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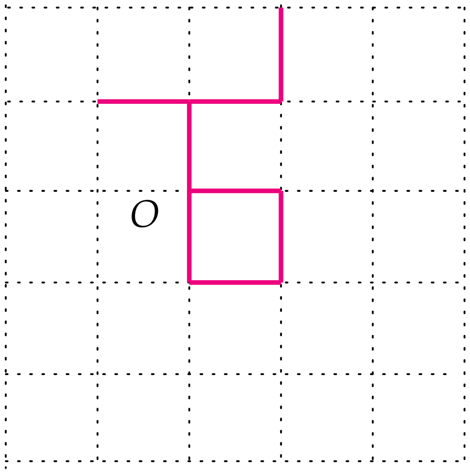
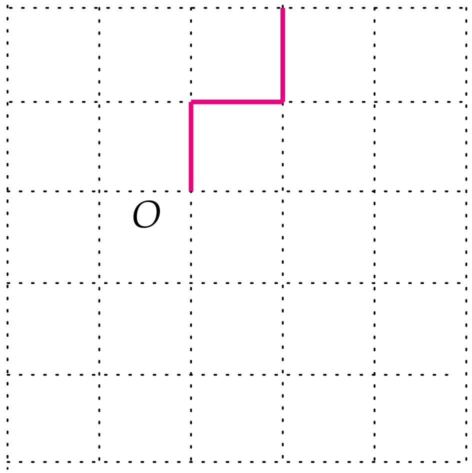


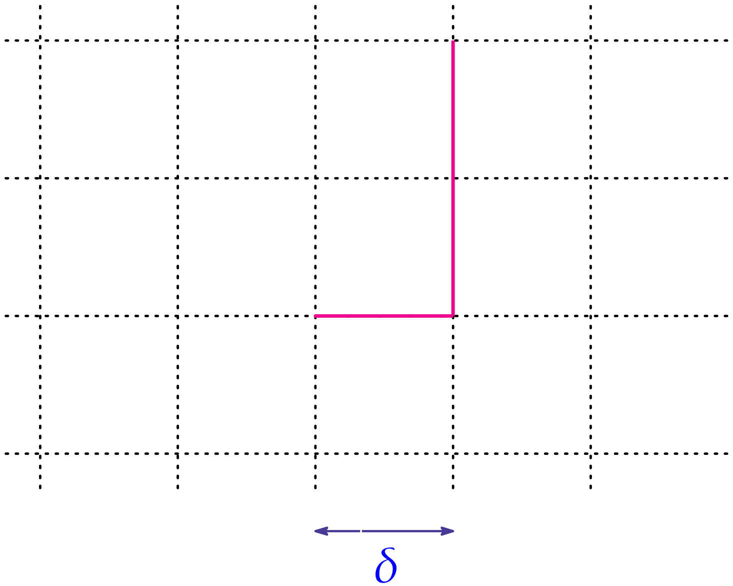


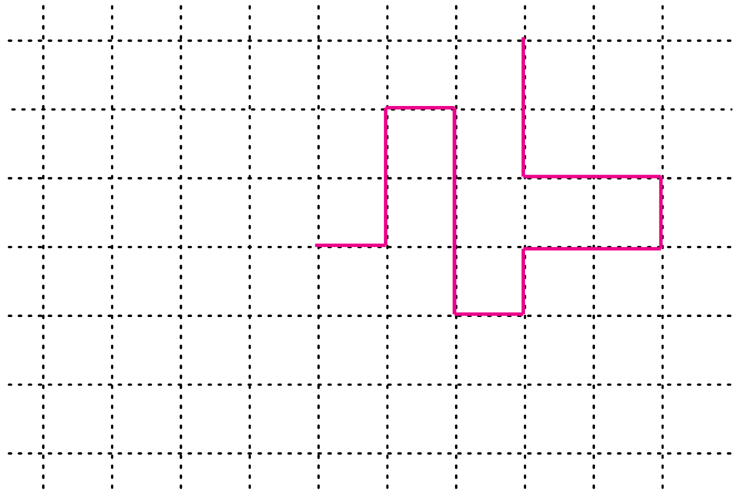
Erase loops in chronological order.

Here we will call it the 'standard' LERW.

0. What I’m going to talk about

Scaling limit of a loop-erased random walk





What kind of process will we have as $\delta \to 0$?

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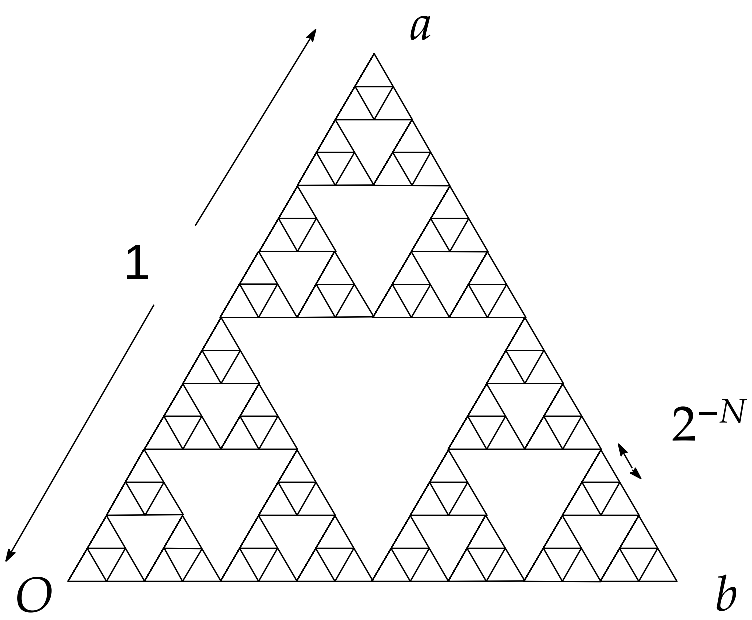
We consider this problem on the pre-Sierpinski gasket



$\delta = 2^{-N} \to 0$?

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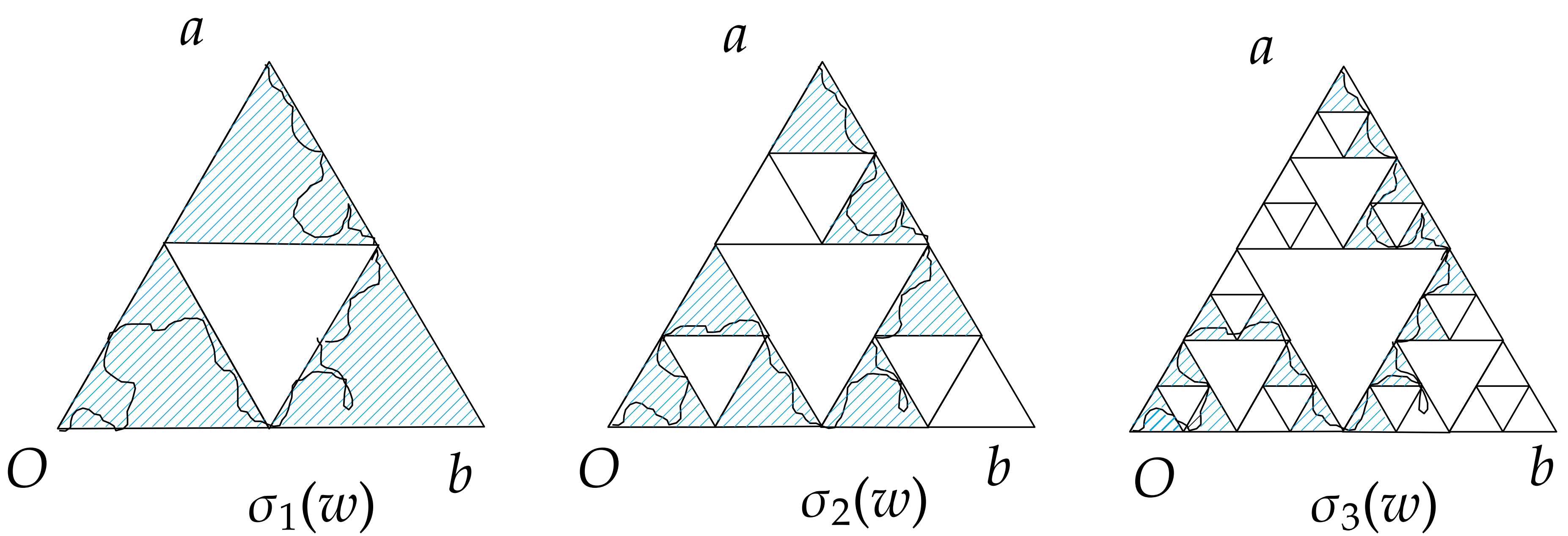
Method : a random fractal approach



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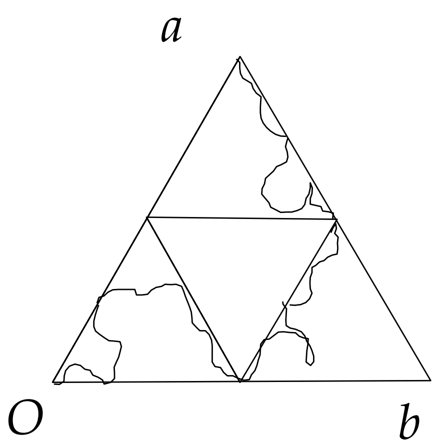
Result : In the scaling limit we get a continuous process, whose path has no self-intersections but has infinitely fine creases ($d_H > 1$).

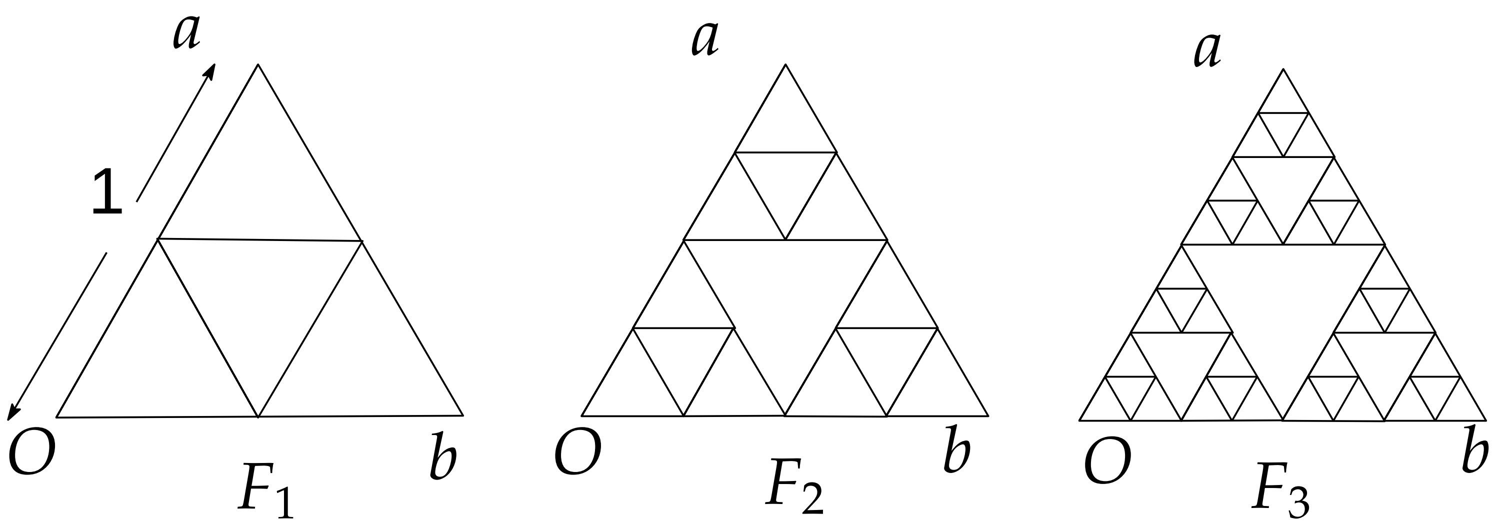


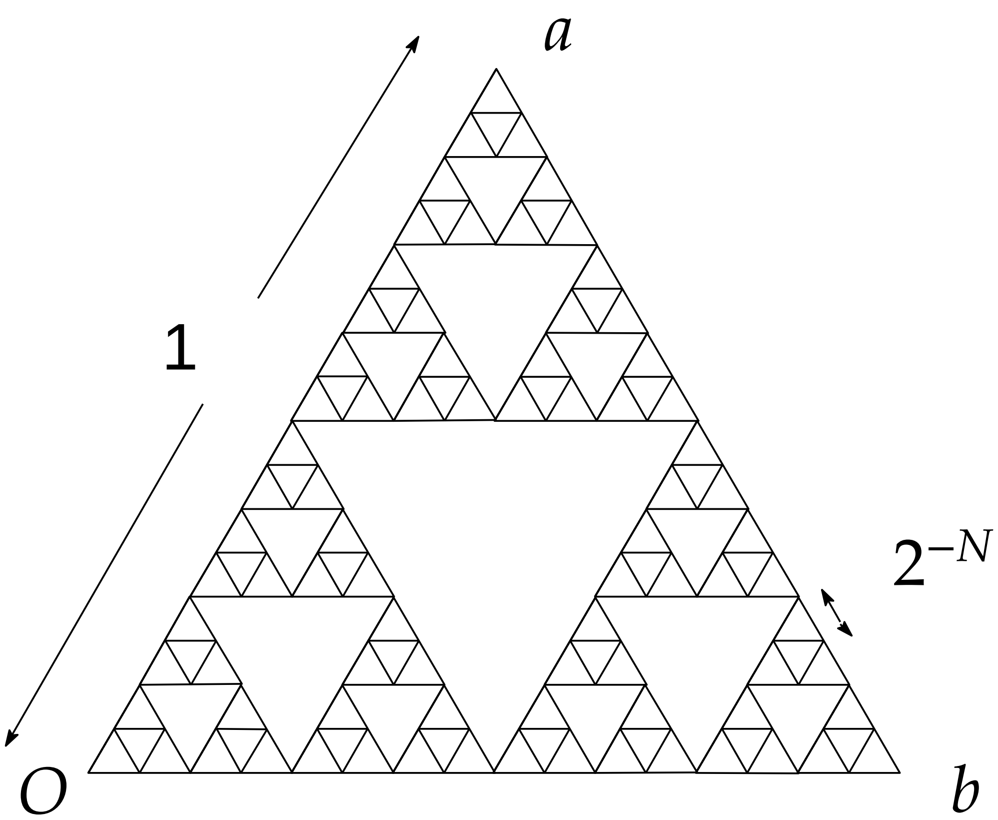
Outline

1. Notations and two basic operations
2. Random fractal approach to LERW
3. Main results (scaling limit)
4. Idea of proof

1-1. The pre-Sierpinski gaskets

The pre-Sierpinski gaskets : A series of finite graphs



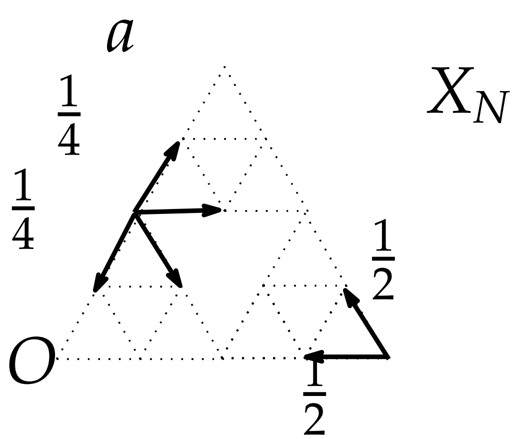


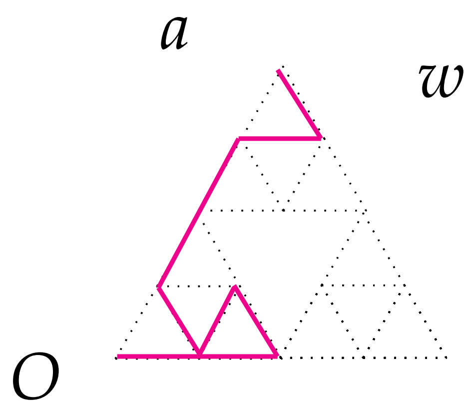
F_N : pre-SG with lattice spacing 2^{-N}

Scaling limit ($N \to \infty$)

$F_N \longrightarrow F$: **Sierpinski gasket** (a fractal)

1-2. Simple random walks on the pre-SG's





$X_N(n)$:SRW on F_N from O to a.

w : a sample path of X_N.

$$X_N(i,\omega) = w(i),\ i = 0, 1, \cdots, \ell(w), w(0) = O, w(\ell(w)) = a$$

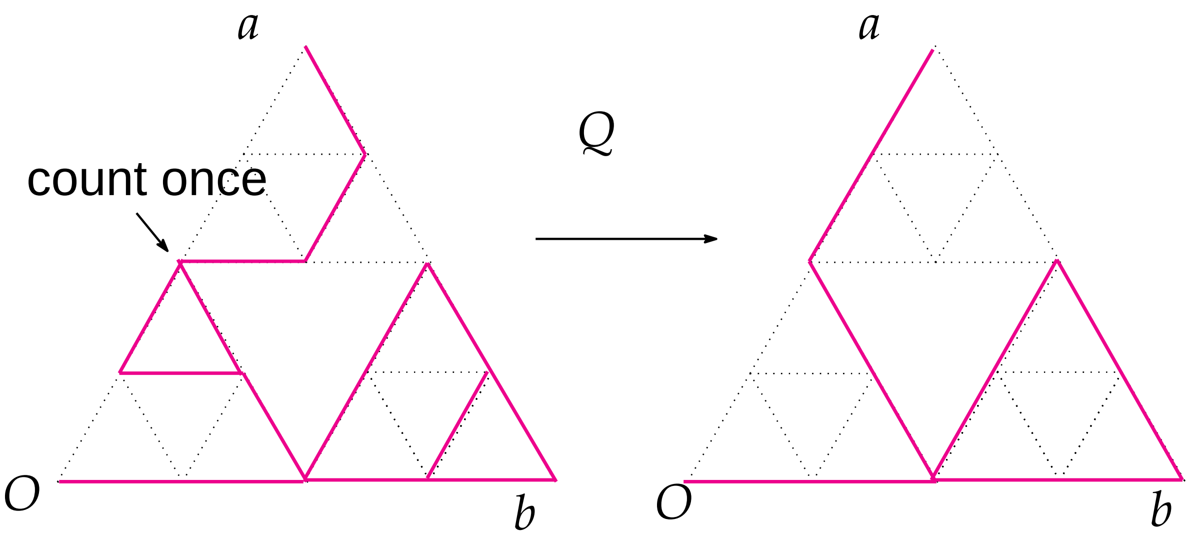
Random fractal approach : We erase loops in an iterative manner using only two kinds of operations, coarse-graining and loop-erasing on F_1.

1-3. Coarse-graining

X_N : SRW on F_N from O to a.

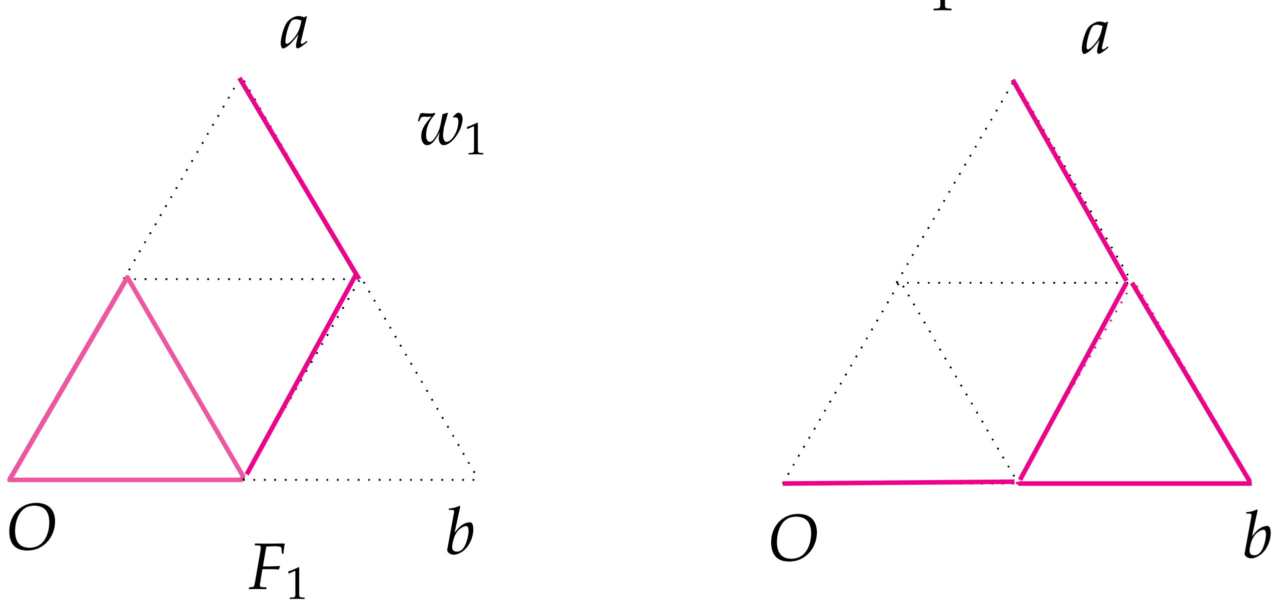
Q : Coarse-graining onto F_1 : Pick up F_1 vertices X_N visits. If it visits a vertex more than once in a row, then count only once.

QX_N is a SRW on F_1 from O to a, that is, X_1.



1-4. Loop-erasing from SRW on F_1

Two conditional SRW on F_1.



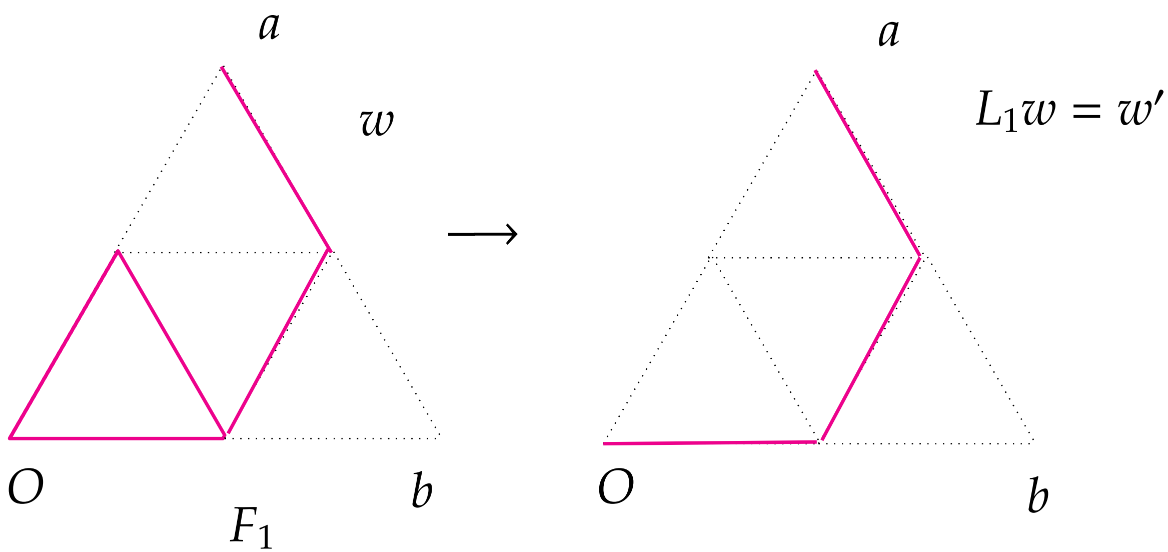
P_1 : the path measure of SRW not via b.

P_1' : the path measure of SRW via b.

For example, $P_1[w_1] = (\frac{1}{2})^2(\frac{1}{4})^4/(\frac{1}{2})$.

Conditioned

L_1 : Loop-erasing operator on random walks on F_1 (chronological).

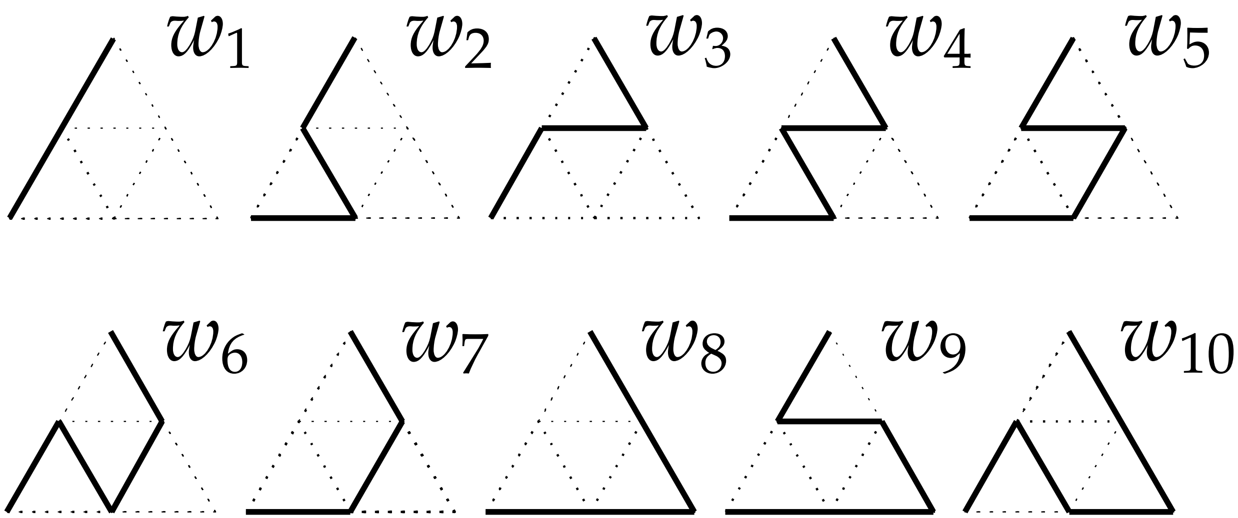


$\hat{P}_1 = P_1 \circ L_1^{-1}$, $\hat{P}'_1 = P'_1 \circ L_1^{-1}$: LERW measures

($\hat{P}_1[w']$ is the probability to get a path w' as a result of loop-erasure.) Infinitely many paths result in a same path.

These probabilities can be calculated directly.

$\hat{P}_1 = P_1 \circ L_1^{-1}$: LERW measure (SRW not via b)

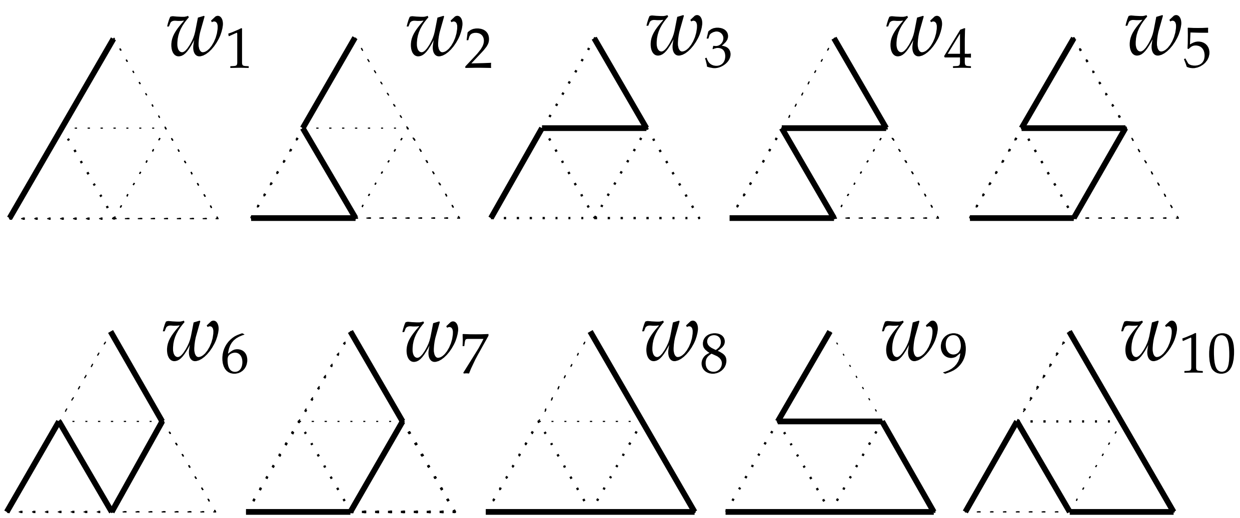


$$\hat{P}_1[w_1] = \frac{1}{2},\ \hat{P}_1[w_2] = \hat{P}_1[w_3] = \frac{2}{15},$$

$$\hat{P}_1[w_4] = \hat{P}_1[w_5] = \hat{P}_1[w_6] = \frac{1}{30},\ \hat{P}_1[w_7] = \frac{2}{15},$$

$$\hat{P}_1[w_i] = 0,\ i = 8, 9, 10.$$

$\hat{P}'_1 = P'_1 \circ L_1^{-1}$: LERW measure (SRW via b)



$$\hat{P}'_1[w_1] = \frac{1}{9},\ \hat{P}'_1[w_2] = \hat{P}'_1[w_3] = \frac{11}{90},$$

$\hat{P}'_1[w_4] = \hat{P}'_1[w_5] = \hat{P}'_1[w_6] = \frac{2}{45}$, ($b$ can be erased)

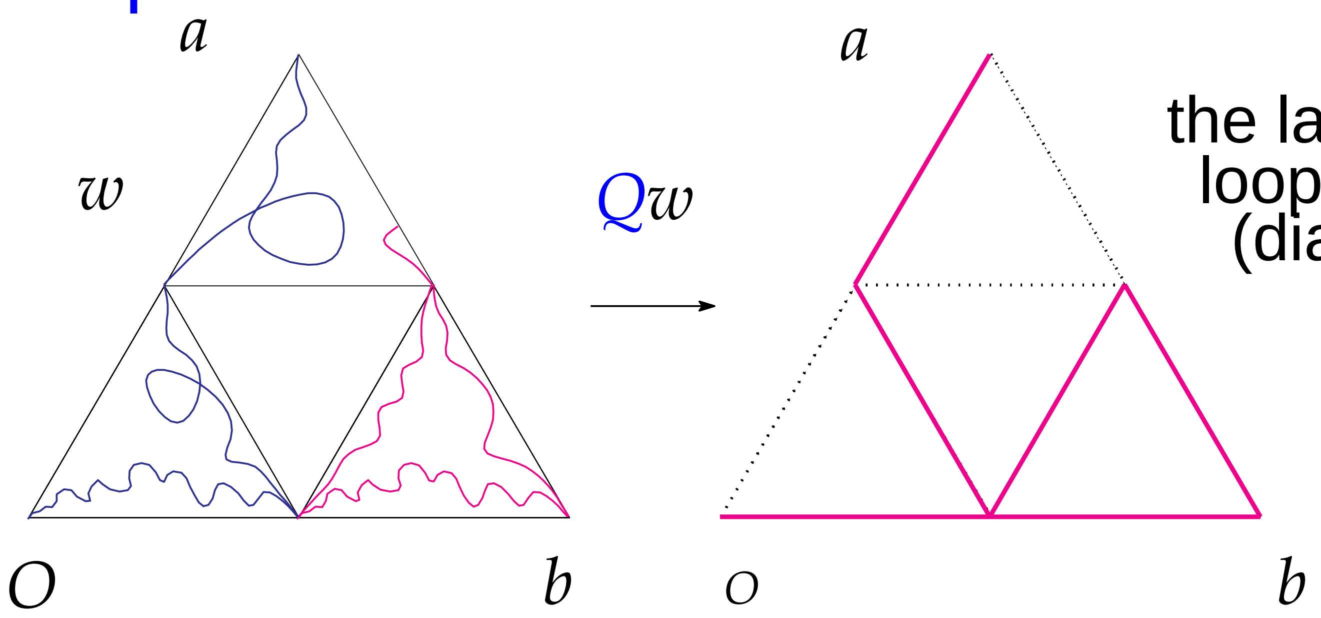
$$\hat{P}'_1[w_7] = \frac{8}{45},\ \hat{P}'_1[w_8] = \frac{2}{9},\ \hat{P}'_1[w_9] = \hat{P}'_1[w_{10}] = \frac{1}{18}.$$

2. Loop-erasing from SRW on F_N

The random fractal approach : erase loops in descending order of size. (not chronologically)

Q and L_1 are enough!

Step 1



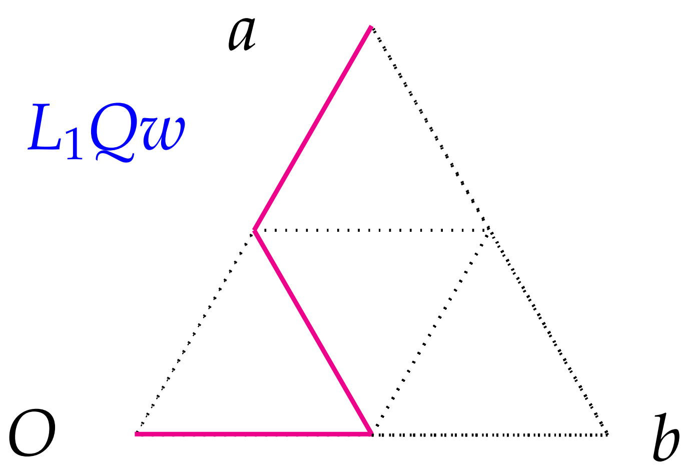
the largest-scale loop (diam $\geq 1/2$)

SRW on F_N
(2^{-N} - lattice)

Coarse-grained walk
(SRW on F_1)

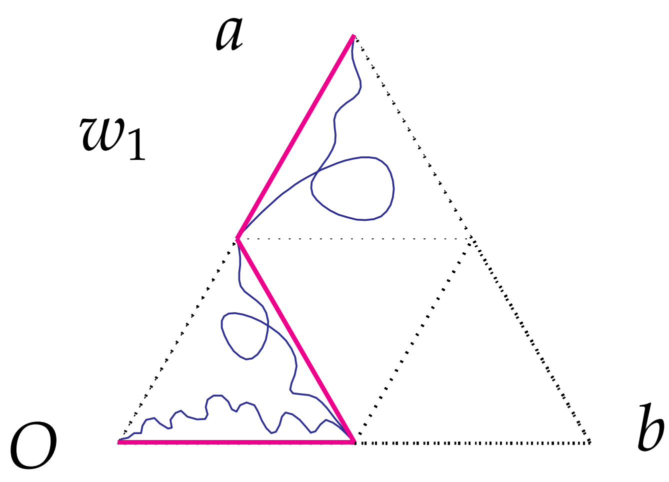
Step 2

Erase loops from Qw

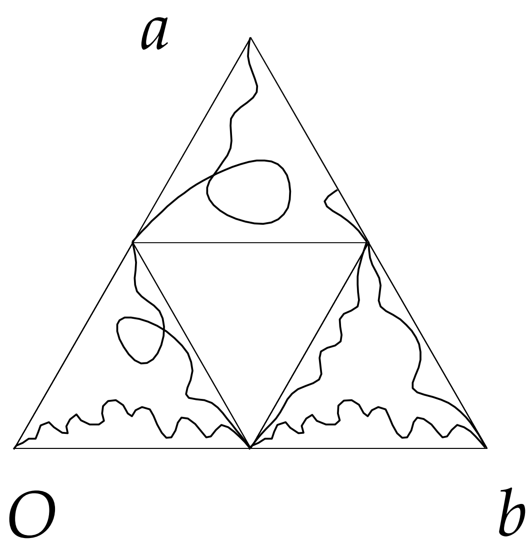


Step 3

Restore fine structure

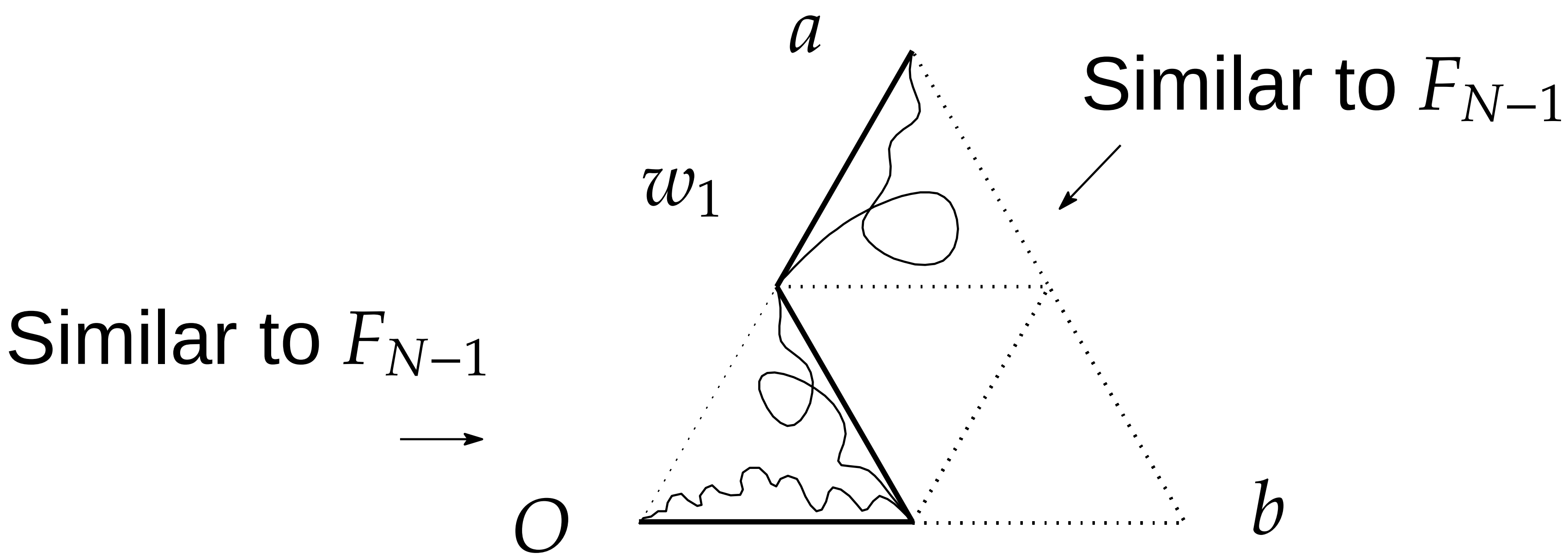


w_1 has no loops with diam $> 2^{-1}$.

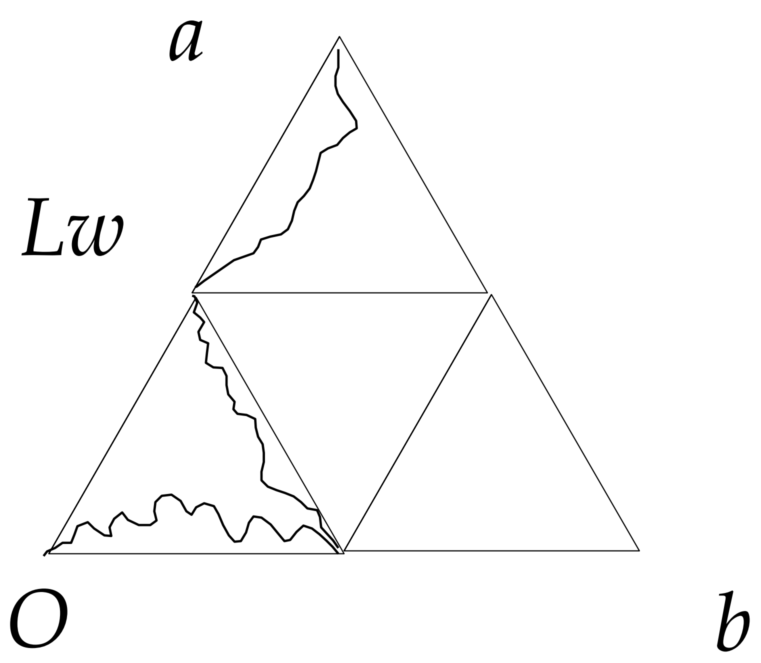


The original path

Each 2^{-1} triangle is similar to F_{N-1}. Apply Step 1–3 to each path segment and erase largest-scale (larger than $1/4$) loops. Repeat until the path has no loops.



Resulting loop-erased path. (After repetition of Q and L_1)



L : Loop-erasing operator

$Y_N = LX_N$: loop-erased random walk on F_N

3. Main results

Theorem 1.

Υ_N : LERW on F_N.
As $N \to \infty$, $\Upsilon_N(\lambda^N \cdot\ )$ converges uniformly in t to a continuous process Υ on the SG a.s.

Theorem 2.

Υ is almost surely self-avoiding.
The path Hausdorff dimension is
$d_{LERW}(\Upsilon([0,\infty))) = \log\lambda / \log 2 = 1.1939\ldots > 1$ a.s.

$$\lambda = \frac{1}{15}(20 + \sqrt{205}) = 2.2878\ldots.$$

Theorem 3.

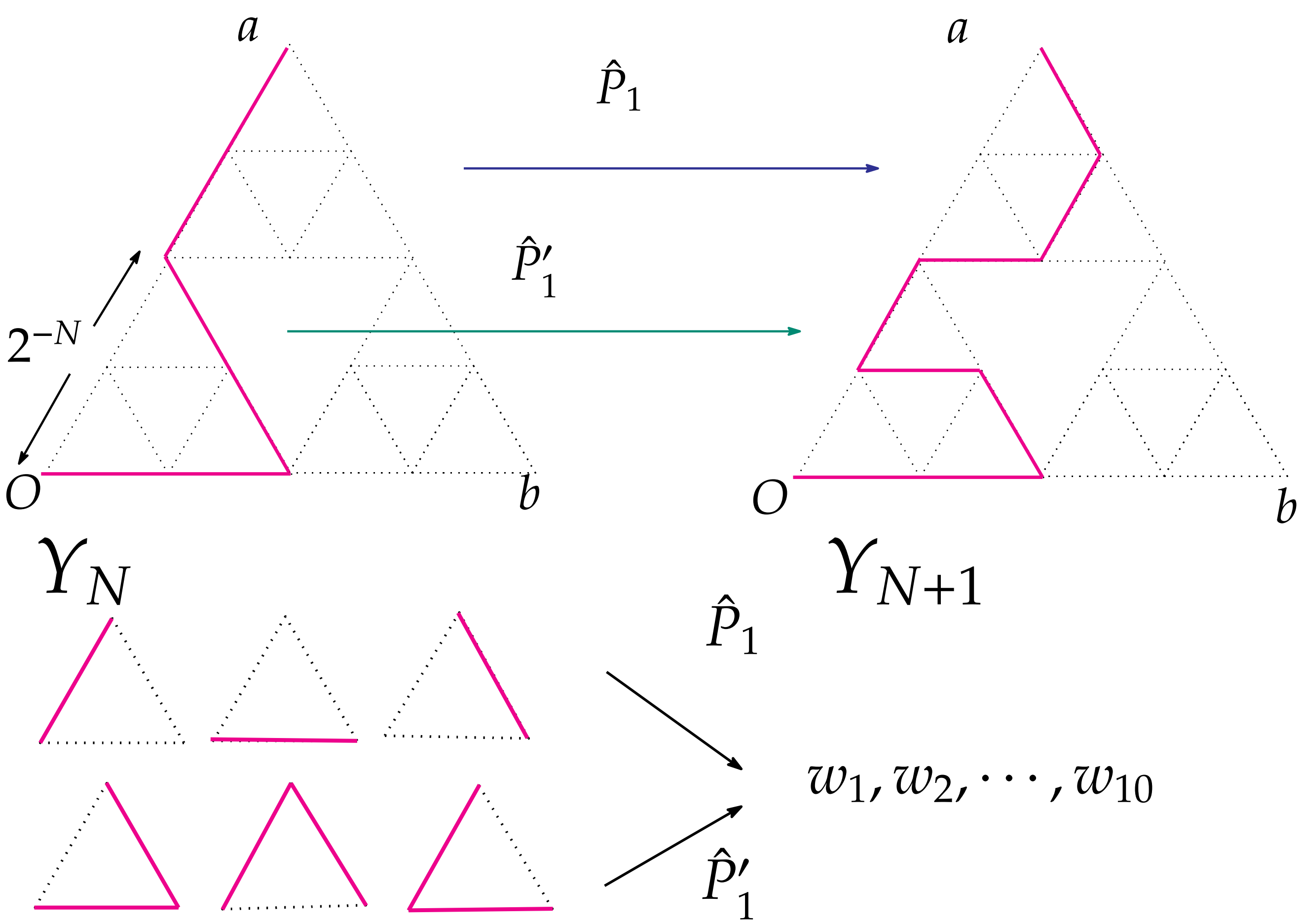
Our LERW has the same distribution as that of the standard LERW (obtained by erasing loops chronologically).

- Note that in general, erasing loops in a different order results in a different stochastic process.

4. Idea of proof

Proposition (branching)

Υ_{N+1} is obtained from Υ_N by the following branching.



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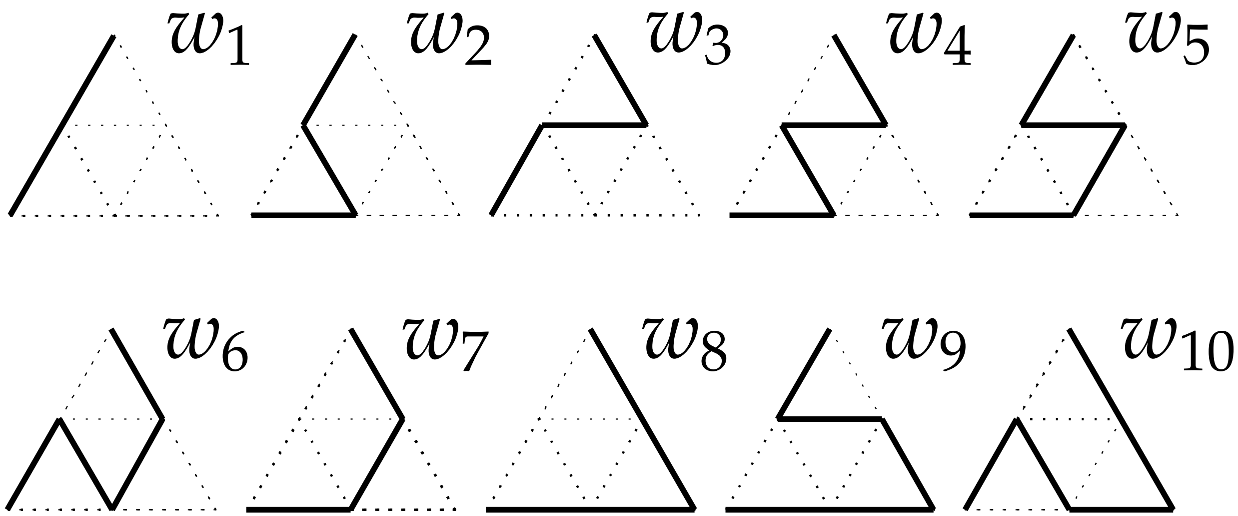
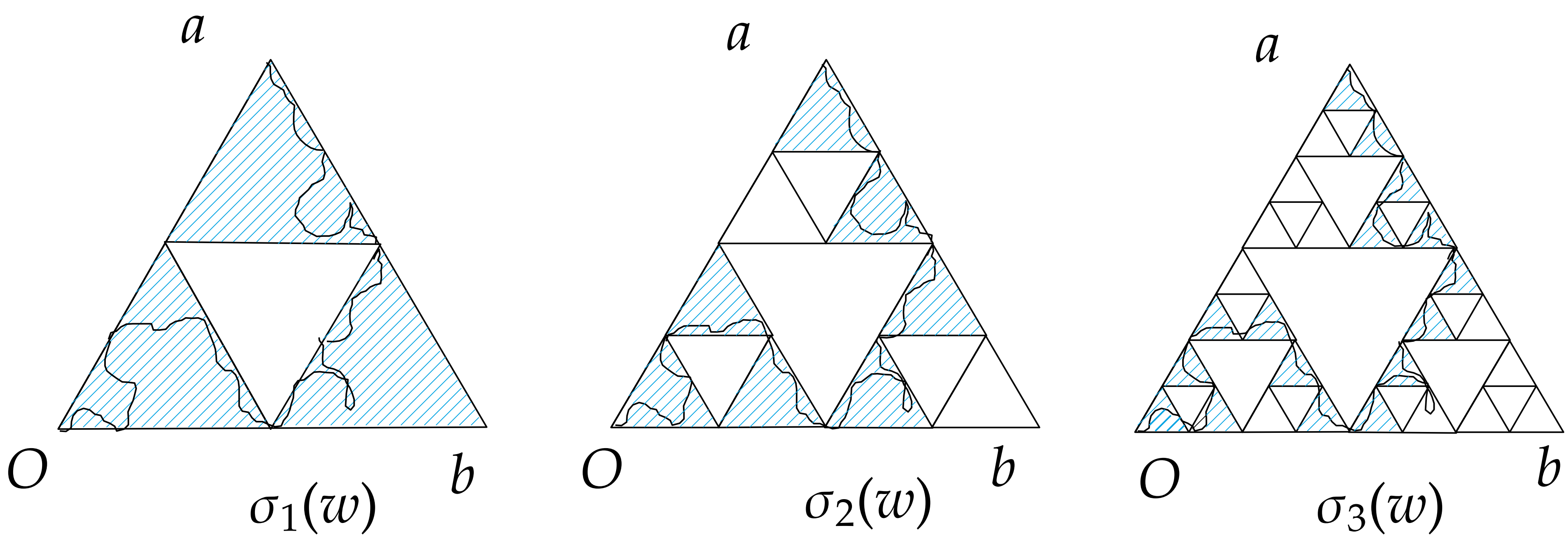
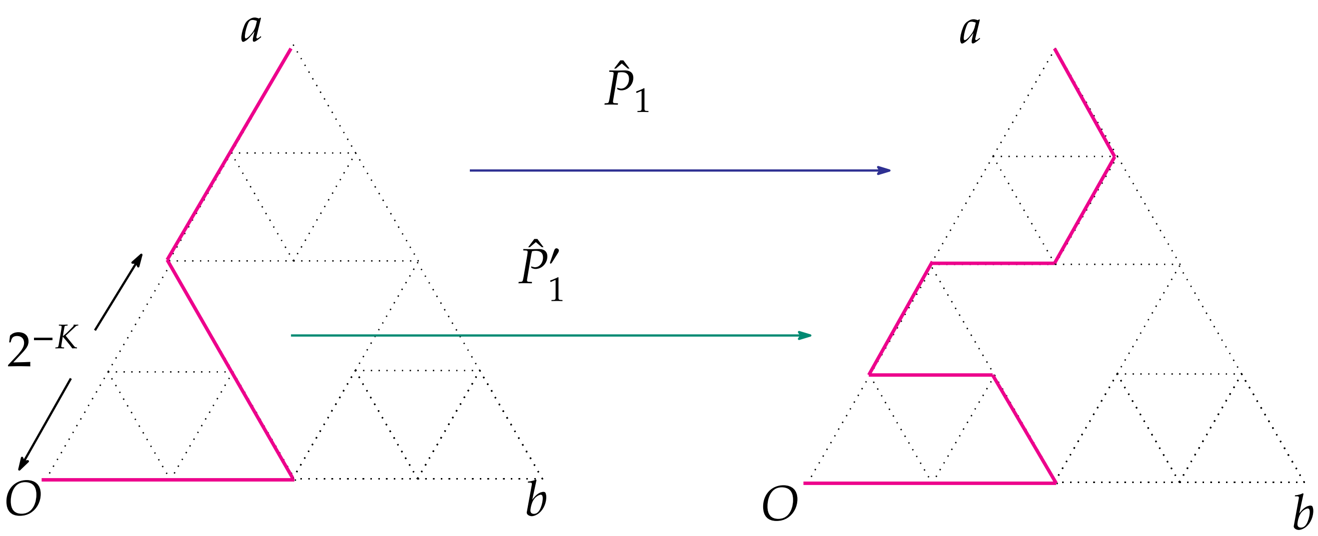


4-1. Path as a closed set

For a path w on the pre-SG or SG, define $\sigma_K(w)$ (K-skeleton of a path w) by the union of (closed) 2^{-K}-triangles w passes through (entance $\neq$ exit) .

$\sigma_1(w) \supset \sigma_2(w) \supset \sigma_3(w) \supset \cdots$

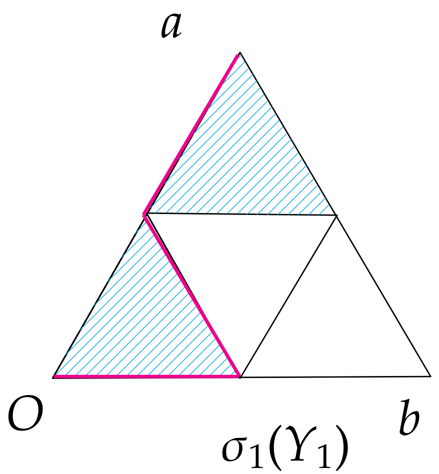


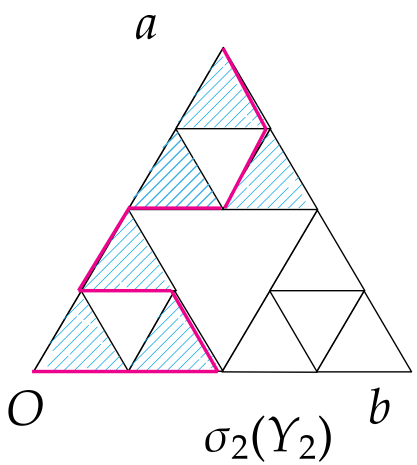


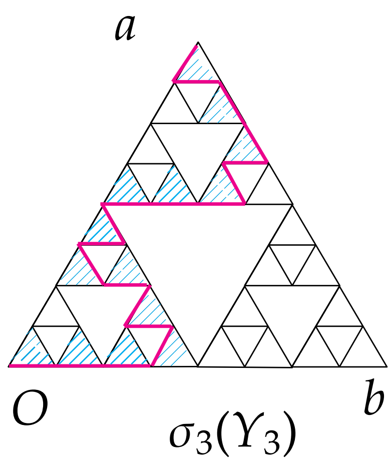
Fix K arbitrarily.

From our construction of branching into finer path,

for any $N \geq K$, $\sigma_K(Y_N) = \sigma_K(Y_K)$, a.s..





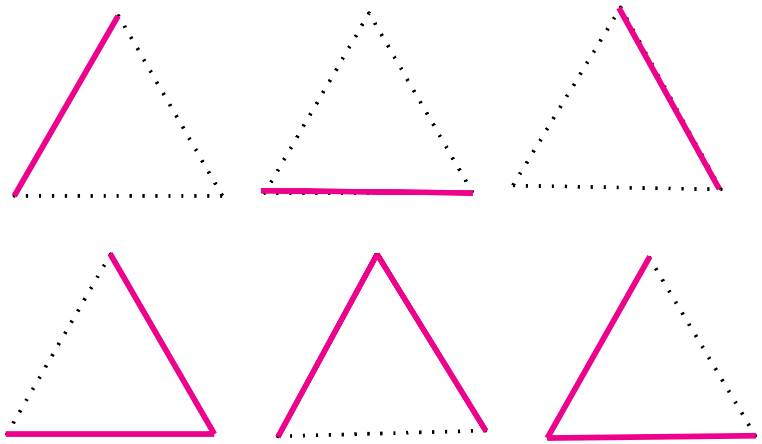


$\sigma_N(Y_N) \supset \sigma_{N+1}(Y_{N+1}) \supset \sigma_{N+2}(Y_{N+2}) \supset \cdots \to \exists\, A$, a.s.

A is a random fractal with $d_H = \log \lambda / \log 2$ a.s.

4-2. Speed on the path

Count the numbers of 2^{-N} -triangles Υ_N passes through:



Type 1

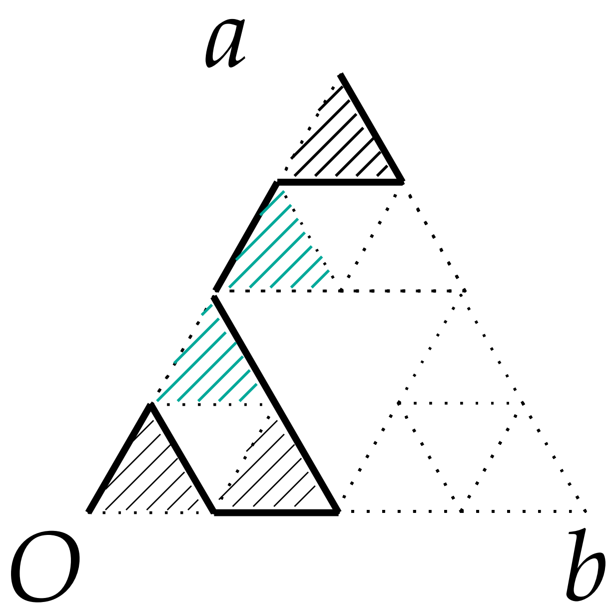
Type 2

$$s_1^N = \sharp\{2^{-N}\text{-triangles of Type 1}\}$$

$$s_2^N = \sharp\{2^{-N}\text{-triangles of Type 2}\}$$

random variables

$s_1^N = \sharp\{\text{Type } 1\}$: two vertices
$s_2^N = \sharp\{\text{Type } 2\}$: three vertices



$$s_1^2 = 2,\ s_2^2 = 3$$

F_2

Number of steps : $\ell^N = s_1^N + 2s_2^N$
(Time taken to go $O \to a$.)

(s_1^N, s_2^N) is a two-type branching process. $\ell^N = s_1^N + 2s_2^N$.

Limit theorem of branching processes

$$\frac{\ell^N}{\lambda^N} \to \exists\, W > 0, \text{ a.s.}$$

$\lambda = \frac{1}{15}(20 + \sqrt{205})$ is chosen so that $E[\frac{\ell^N}{\lambda^N}]$ is independent of N.

This theorem garantees the convergence of crossing time of any triangle.

Summary

• The existence of the scaling limit of LERW.

• The path of the limiting process has infinitely fine creases, while having no self-intersection.

• Our model and the standard LERW are the same.

• Shinoda, Teufl and Wagner obtained the scaling limit independently, by a different method (using uniform spanning trees).

References

- G. Lawler, *Intersections of Random Walks*, Birkhäuser, (LERW on $\mathbb{Z}^d$)

- M. Shinoda, E. Teufl, S. Wagner *Uniform spanning trees on Sierpinski graphs*, arXiv:1305.5114

- Hattori, Mizuno, *Loop-erased random walk on the Sierpinski gasket*, SPA 124 (2014) 566–585

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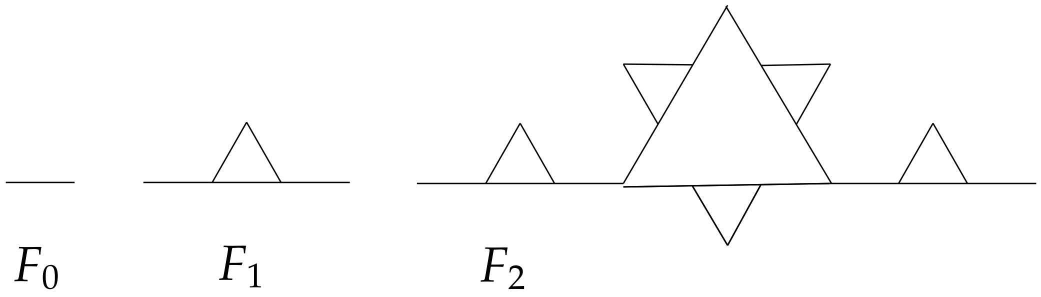
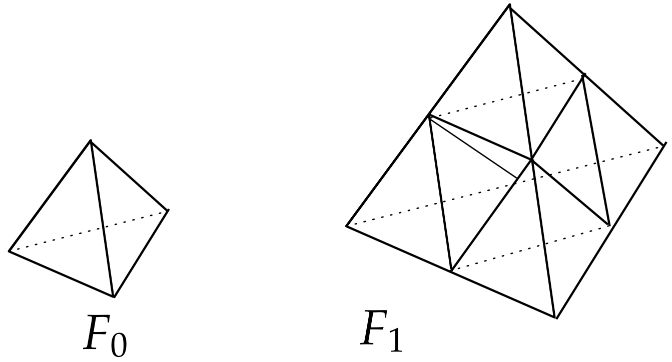
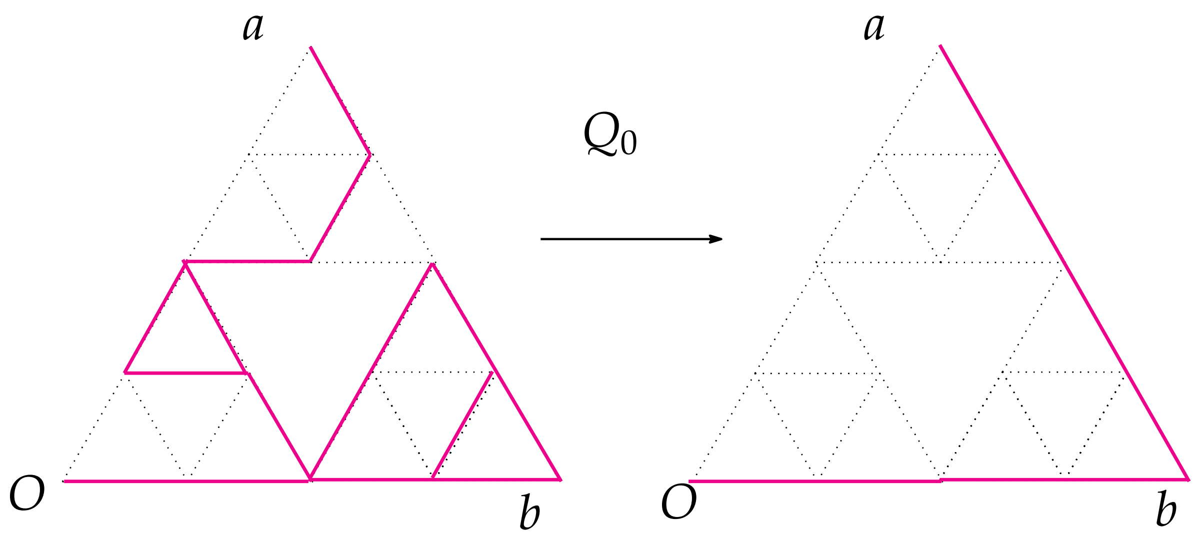

F_0
F_1
F_2


F_0
F_1

Q_0 : Coarse-graining onto F_0



Above, $((Q_0 X_N)(0), (Q_0 X_N)(1), (Q_0 X_N)(2)) = (O, b, a)$.

Conditional path measures

P_1 : $((Q_0 X_N)(0), (Q_0 X_N)(1)) = (O, a)$.

P'_1 : $((Q_0 X_N)(0), (Q_0 X_N)(1), (Q_0 X_N)(2)) = (O, b, a)$.

• The LERW belongs to a different universarity class from SAW studied earlier (H., T. Hattori, Kusuoka).

$$d_{LERW} = \frac{\log(20 + \sqrt{205})/15}{\log 2} = 1.1939\ldots$$

$$d_{SAW} = \frac{\log(7 - \sqrt{5})/2}{\log 2} = 1.2521\ldots$$

Generating functions

$\hat{W}_N$: The set of loopless paths on F_N from O to a,

$\hat{P}_N = P_N \circ L^{-1}$, $\hat{P}'_N = P'_N \circ L^{-1}$: LERW path measures

$$\Phi_N(x, y) = \sum_{w \in \hat{W}_N} \hat{P}_N(w)\, x^{s_1(w)}\, y^{s_2(w)},$$

$$\Theta_N(x, y) = \sum_{w \in \hat{W}_N} \hat{P}'_N(w)\, x^{s_1(w)}\, y^{s_2(w)}, \quad x, y \geq 0.$$

Recursions

Proposition.

The recursion relations:

$$\Phi_{N+1}(x,y) = \Phi_1(\Phi_N(x,y), \Theta_N(x,y)).$$

$$\Theta_{N+1}(x,y) = \Theta_1(\Phi_N(x,y), \Theta_N(x,y)), \quad N \in \mathbb{N}.$$

$$\Phi_1(x,y) = \frac{1}{30}(15x^2 + 8xy + y^2 + 2x^2y + 4x^3).$$

$$\Theta_1(x,y) = \frac{1}{45}(5x^2 + 11xy + 2y^2 + 14x^2y + 8x^3 + 5xy^2).$$

Mean matrix of the number of triangles

$$\mathbf{M} = \begin{bmatrix} \frac{\partial}{\partial x}\Phi_1(1,1) & \frac{\partial}{\partial y}\Phi_1(1,1) \\ \frac{\partial}{\partial x}\Theta_1(1,1) & \frac{\partial}{\partial y}\Theta_1(1,1) \end{bmatrix} = \begin{bmatrix} \frac{9}{5} & \frac{2}{5} \\ \frac{26}{15} & \frac{13}{15} \end{bmatrix}$$

The larger eigenvalue

$$\lambda = \frac{1}{15}(20 + \sqrt{205}) = 2.2878\ldots.$$

The average steps from O to a on F_N $\sim \lambda^N$ $(N \to \infty)$

$(\ell(w) = s_1(w) + 2s_2(w))$ n steps $\to$ time n

Need an appropriate time-scaling $\longrightarrow X_N(\lambda^N \cdot \ )$

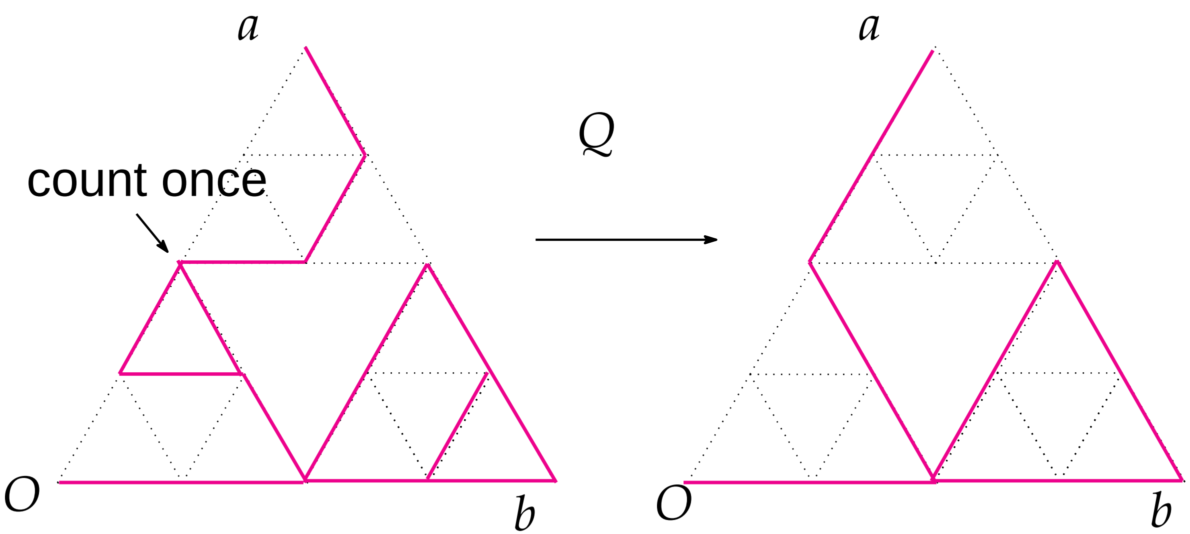
X_N : LERW on F_N

Loop-erasing from SRW on F_N

Erasing-larger-scale-loops-first rule. (not chronologically)

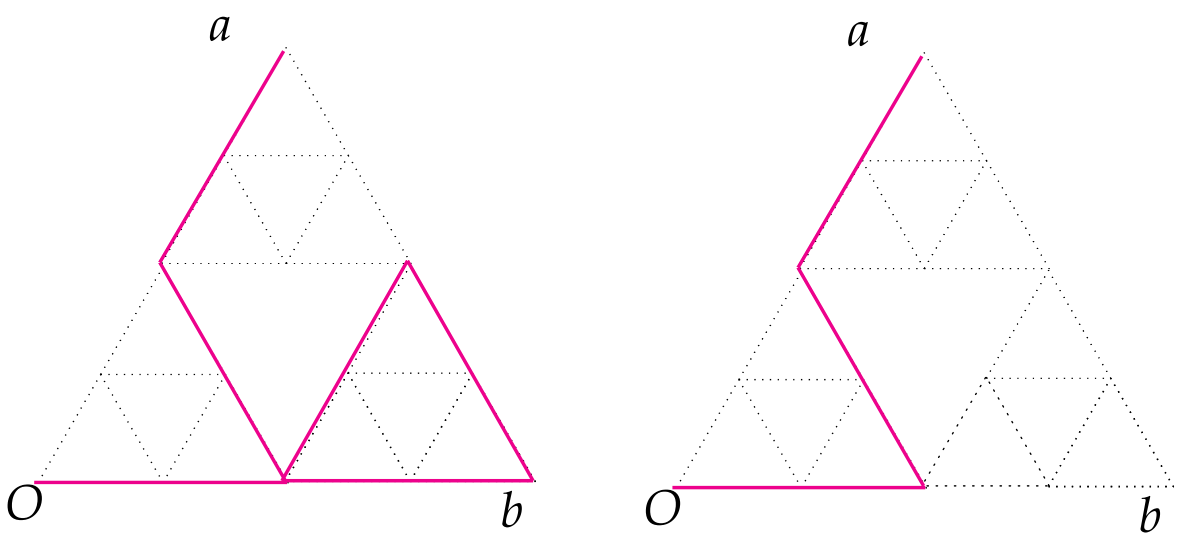
Q and L_1 are enough for our loop-erasing (random fractal approach).

Step 1: Coarse-grain onto F_1

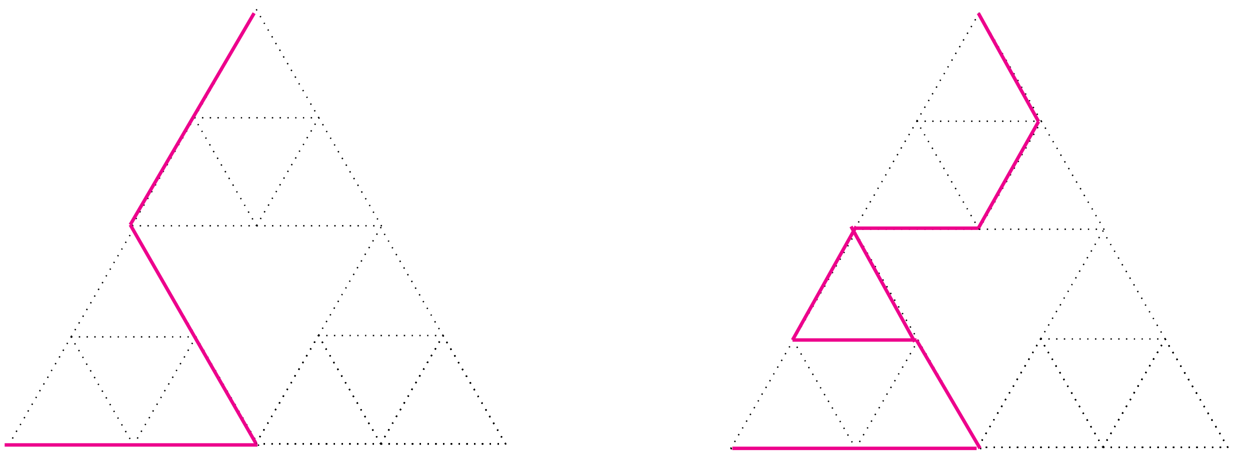


Note that if w has a loop with diameter larger than $1/2$, then $Q_1 w$ has a loop.

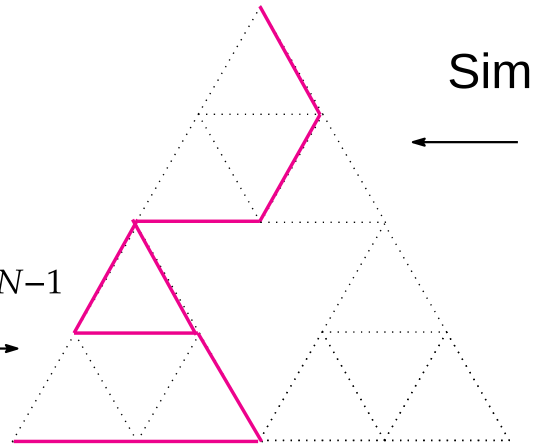
Step 2: Erase loop from $Q_1 w$.



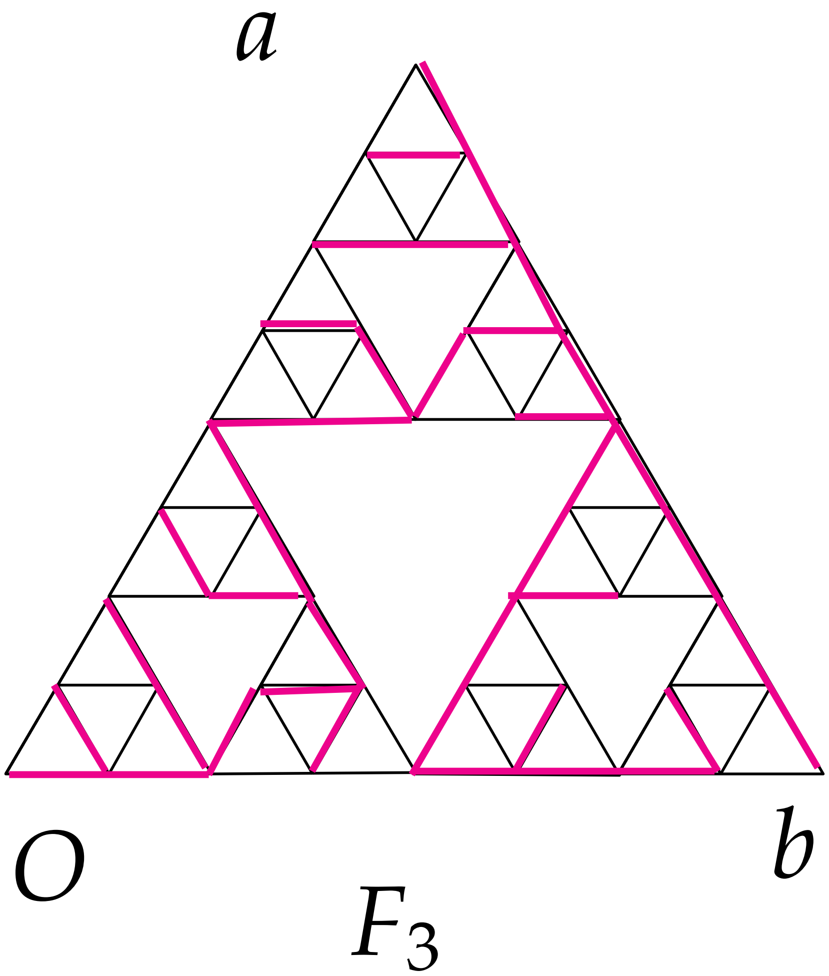
Step 3: Give back finer structure to $LQ_1 w$.



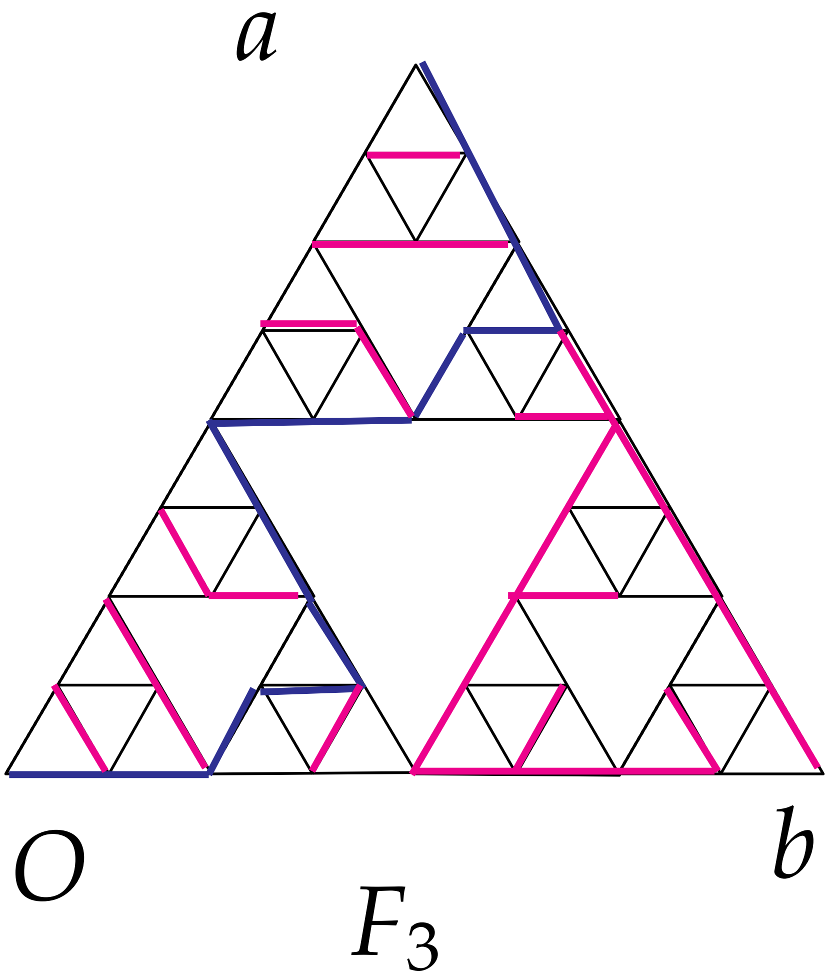
Now all loops with diameter larger than $1/2$, are gone.



Each 2^{-1} triangles are similar to F_{N-1}. Apply the same procedure to each path segment and erase the largest-scale loops (larger than $1/4$) loops. Repeat until we have no loops. (Repetition of Q_1 and L.)



A spanning tree on F_3



A spanning tree on F_3

The unique path $O \to a$