Self-repelling Walk on the Sierpiński Gasket

Ben HAMBLY
Mathematical Institute, University of Oxford,
24-29 St Giles, Oxford OX1 3LB, UK,
e-mail address: hambly@maths.ox.ac.uk

Kumiko HATTORI
Department of Mathematical Sciences, Shinshu University,
Asahi, Matsumoto, 390-8621, Japan,
e-mail address: hattori@gipac.shinshu-u.ac.jp

Tetsuya HATTORI
Graduate School of Mathematics, Nagoya University,
Chikusa-ku, Nagoya, 464-8602, Japan,
e-mail address: hattori@math.nagoya-u.ac.jp

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Abstract

We construct a one-parameter family of self-repelling processes on the Sierpiński gasket, by taking continuum limits of self-repelling walks on the pre-Sierpiński gaskets. We prove that our model interpolates between the Brownian motion and the self-avoiding process on the Sierpiński gasket. Namely, we prove that the process is continuous in the parameter in the sense of convergence in law, and that the order of Hölder continuity of the sample paths is also continuous in the parameter. We also establish a law of the iterated logarithm for the self-repelling process. Finally we show that this approach yields a new class of one-dimensional self-repelling processes.

1 Introduction.

In this paper we construct and study a one-parameter family of self-repelling processes on the Sierpiński gasket by taking continuum limits of self-repelling random walks on the pre-Sierpiński gaskets.

The continuum limits for the following two cases are known on the pre-Sierpiński gaskets:

(1) The simple random walk, whose continuum limit is the Brownian motion on the Sierpiński gasket [2, 8, 19].

(2) The self-avoiding path, whose continuum limit is a self-avoiding process on the Sierpiński gasket [12, 13, 15].

Note that the self-avoiding process on the Sierpiński gasket has a non-trivial distribution [12, 13, 15] (in contrast to the one-dimensional self-avoiding process, which is a deterministic linear motion).

Our family of processes is parametrized by $u \in [0, 1]$ such that the Brownian motion corresponds to $u = 1$, and the self-avoiding process corresponds to $u = 0$. The processes corresponding to $0 < u < 1$ interpolate between the two extreme cases in the sense that the path measure P^u (the image measure of the process defined on a space of continuous functions on the Sierpiński gasket) converges weakly as $u \to u_0$ for any $0 \leqq u_0 \leqq 1$, and that the scaling exponent γ for the 'speed' of the process is continuous in $u \in [0, 1]$.

The initial work on self-avoiding and weakly self-avoiding (self-repelling) random walk arises from defining models for polymers. The classical problem is to define a path measure on $\mathbb{R}^d$ in which self intersections are penalized by an exponential weighting factor and then study the almost sure behaviour of the paths under this measure. The one dimensional case is reasonably well understood as well as for $d > 4$ but in two and three dimensions there are still major open problems.

An important property is the behaviour of the end to end length of the polymer. This is captured in an exponent for the speed of the walk, (the reciprocal of the walk dimension) γ, which can be defined by $\gamma = \lim_{n\to\infty} \log|X_n|/\log(n)$. On the one dimensional integer lattice [5, 9, 18] proved that there is 'ballistic motion' in that $\gamma = 1$. The models proposed in [21, 22] are consistent with $\gamma = 2/3, \gamma = 1/2$ respectively. In these cases the exponents γ are independent of the self-repelling factor. A model which continuously interpolates between $1/2 \leqq \gamma \leqq 2/3$ is given in [20].

All these models obtain the self-repelling property by introducing weights depending on the number of returns to bonds [20, 21, 22] or sites [5, 9, 18]. For a recent review of the site case see [17]. For some models it is possible to construct continuum limit processes, and [6] constructs such a process on $\mathbb{R}$ in the diffusive phase ($\gamma = 1/2$). In [23] a continuous self-repelling process is constructed in the case $\gamma = 2/3$ and many of its path properties are examined.

Our approach is different in that we introduce a parameter u which allows us to interpolate between the simple random walk and a self-avoiding walk. Our path is weighted according to a revisiting factor, which counts visits to 'higher level' points, and a reversing factor, which counts back tracks. (By reversing we mean, for the Sierpiński gasket, that the path remains within the same triangle, not necessarily going back the way it came.) We will take a continuum limit of the random walks to obtain a self-repelling process. It is known that the scaling exponent γ is different for the Brownian motion and the self-avoiding process. We prove that γ is continuous for $0 \leqq u \leqq 1$. Since all the processes we consider are self-similar, this should imply that various exponents of the sample paths of the processes are also continuous in u in our model, and interpolate between those of the Brownian motion and the self-avoiding process on the Sierpiński gasket.

We note that our parameter does not directly count the number of returns to bonds or sites. Therefore, our construction gives an alternative model for self-repelling walks on $\mathbb{Z}$. We can construct a one-parameter family of self-repelling processes on $\mathbb{R}$ by taking continuum limits of these self-repelling random walks, and the resulting family of processes continuously interpolates the exponent γ between that of the one-dimensional Brownian motion and the one-dimensional self-avoiding process (deterministic linear motion).

The structure of the paper is as follows. We will begin with a sequence of random walks on graph approximations to the Sierpinski gasket and show in Section 2 that there is a continuum limit process for each value of the interpolating parameter. In Section 3 we prove that the processes are continuous in the interpolating parameter. We also note that though we construct our continuum limit processes on a finite Sierpiński gasket, the extension of our processes to the (infinite) Sierpiński gasket in a self-similar way apparently poses no difficulties. The corresponding results, with the expected self-similarity, will imply that the exponents of mean square displacement lie between the value of the Brownian motion and the self-avoiding process for our model, and that the value is a continuous function of the parameter. Thus it is natural to consider our model as a family of self-repelling processes which interpolates between the Brownian motion and the self-avoiding process on the Sierpiński gasket.

In Section 4 we will discuss the path properties of our self-repelling process. In particular we show that γ controls the mean square displacement and prove a law of the iterated logarithm. A crucial part of the proof is that there is a supercritical branching process which describes the path. In what follows we will mainly work on the more difficult case of the Sierpiński gasket. In the final section we will summarize the basic ingredients (and the main differences from the Sierpiński gasket case) of the construction of the corresponding self-repelling processes on $\mathbb{R}$.

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2 Construction of the processes.

The pre-Sierpiński gaskets and the Sierpiński gasket are defined as follows. Let $O = (0,0)$, $a = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $b = (1,0)$, and let F'_0 be the set of all the points on the vertices and edges of $\triangle Oab$. We define a sequence of sets F'_0, F'_1, F'_2, ... , inductively by

$$F'_{n+1} = \frac{1}{2}F'_n \cup \frac{1}{2}(F'_n + a) \cup \frac{1}{2}(F'_n + b), \qquad n = 0, 1, 2, \dots ,$$

where $A+a=\{x+a\ :\ x\in A\}$ and $kA=\{kx\ :\ x\in A\}$. Let

$$F_n = F'_n \cup (F'_n - b).$$

We call F_n's the (finite) pre-Sierpiński gaskets, and $F = cl(\cup_{n=0}^{\infty} F_n)$ the (finite) Sierpiński gasket. We denote the set of vertices in F_n by G_n. Let us denote by $\mathcal{T}_n$ the set of all the closed triangles in $\mathbb{R}^2$ that are the translations of $2^{-n}\triangle Oab$ (without rotation) and whose vertices are in G_n.

Let

$$C = \{w \in C([0,\infty)\to F)\ :\ w(0)=O,\ \lim_{t\to\infty} w(t) = a\}.$$

Let $|x-y|$, $x,y\in\mathbb{R}^2$, denote the Euclidean distance. C is a complete separable metric space with the metric

$$d(u,v) = \sup_{t\in[0,\infty)} |u(t)-v(t)|,\ u,v\in C.$$

We define the 'hitting times,' $T_i^k : C \to \mathbb{R}_+\cup\{\infty\}$, $k,i\in\mathbb{Z}_+$, as follows. Let $T_0^k(w)=0$, and by induction, for $i\geq 1$, let

$$T_i^k(w) = \inf\{t > T_{i-1}^k(w)\ :\ w(t)\in G_k \setminus \{w(T_{i-1}^k(w))\}\},$$

if the right hand side is finite, otherwise, $T_i^k(w)=\infty$. T_i^k is the time when the path w hits a vertex of G_k for the i-th time under the condition that if w hits the same element of G_k more than once in a row, we consider it 'once'. Writing $w(\infty)=a$, and noting that $w(t)\to a$ as $t\to\infty$, we obtain a finite sequence $\{T_i^k\}_{i=1,\dots,M}$ such that $w(T_M^k(w))=a$ and $w(T_{M-1}^k(w))\neq a$. Let $S_i^k(w)=T_i^k(w)-T_{i-1}^k(w)$.

For $n\in\mathbb{Z}_+$, we define a 'decimation' map $Q_n : C\to C$ by setting

$$(Q_n w)(i) = w(T_i^n(w)),$$

for $i=0,1,2,\dots,M$, where M is as above, and by using linear interpolation

$$(Q_n w)(t) = \begin{cases} (i+1-t)\ (Q_n w)(i) + (t-i)\ (Q_n w)(i+1), & i \leqq t < i+1,\ i=0,1,2,\dots,M-1,\\ a, & t\geq M.\end{cases}$$

Note that

(2.1)
$$Q_k \circ Q_n = Q_k, \quad \text{if} \quad k \leqq n.$$

Let us denote by W_n the set of continuous functions $w:[0,\infty)\to F_n$ such that there exists $L(w)\in\mathbb{N}$ for which

$$\begin{array}{ll} w(0)=O, & \\ w(t)=a, & t\geq L(w),\\ w(t)\notin G_0\setminus\{O\}, & t<L(w),\\ |w(i)-w(i+1)|=1, & i=0,\cdots,L(w)-1,\\ \overline{w(i)w(i+1)}\subset F_n, & i=0,\cdots,L(w)-1,\\ w(t)=(i+1-t)w(i)+(t-i)w(i+1), & i\leqq t<i+1,\quad i=0,1,2,\cdots. \end{array}$$

These are all the paths from 0 to a which remain in a pair of triangles about 0 and first exit at a.

Also we denote by $W_n^{(1)}$ the set of continuous functions $w:[0,\infty)\to F'_n$ such that there exists $L_1(w)\in\mathbb{N}$ for which

$$\begin{array}{ll} w(0)=O, & \\ w(t)=O, & t\geq L_1(w),\\ w(t)\notin G_0, & 0<t<L_1(w),\\ |w(i)-w(i+1)|=1, & i=0,\cdots,L_1(w)-1,\\ \overline{w(i)w(i+1)}\subset F_n, & i=0,\cdots,L_1(w)-1,\\ w(t)=(i+1-t)w(i)+(t-i)w(i+1), & i\leqq t<i+1,\quad i=0,1,2,\cdots. \end{array}$$

These are the excursions from 0 which do not reach a or b.

Finally we denote by $W_n^{(2)}$ the set of continuous functions $w:[0,\infty)\to F_n$ such that there exists $L_2(w)\in\mathbb{N}$ for which

$$\begin{array}{ll} w(0)=O, & \\ w(t)=a, & t\geq L_2(w),\\ w(t)\notin G_0, & t<L_2(w),\\ |w(i)-w(i+1)|=1, & i=0,\cdots,L_2(w)-1,\\ \overline{w(i)w(i+1)}\subset F_n, & i=0,\cdots,L_2(w)-1,\\ w(t)=(i+1-t)w(i)+(t-i)w(i+1), & i\leqq t<i+1,\quad i=0,1,2,\cdots. \end{array}$$

These paths are the excursions from 0 which exit at a.

We call $L(w)$, $L_1(w)$ and $L_2(w)$ the length of the path w.

W_n and $W_n^{(2)}$ are subsets of C. We define T_i^m's and Q_m's also on $W_n^{(1)}$ analogously to the definitions on C.

Each $w \in W_n$ makes a polygonal curve on F_n. For $w \in \bigcup_{n\geq k}(W_n \cup W_n^{(1)} \cup W_n^{(2)})$, define the reversing number $N_k(w)$ and the returning number $M_k(w)$ for level k by

$$\begin{aligned} N_k(\ell)(w) &= \sharp\{T_{\ell-1}^{k-1} < i < T_\ell^{k-1} \ : \ \overrightarrow{(Q_k w)(i-1)(Q_k w)(i)} \cdot \overrightarrow{(Q_k w)(i)(Q_k w)(i+1)} < 0, \\ &\quad (Q_k w)(i) \neq w(T_{\ell-1}^{k-1}(w)) \ \}, \end{aligned}$$

where $\vec{a} \cdot \vec{b}$ denotes the inner product of $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^2$, and

$$\begin{aligned} M_k(\ell)(w) &= \sharp\{T_{\ell-1}^{k-1} < i < T_\ell^{k-1} \ : \ (Q_k w)(i) = w(T_{\ell-1}^{k-1}(w))\}, \\ &\qquad \ell = 1, \cdots, L(Q_{k-1}w), \\ N_k(w) &= \sum_{\ell=1}^{L(Q_{k-1}w)} N_k(\ell)(w), \\ M_k(w) &= \sum_{\ell=1}^{L(Q_{k-1}w)} M_k(\ell)(w). \end{aligned}$$

Thus $N_k(w)$ counts the number of times the path w on G_k makes two steps within the same triangle and $M_k(w)$, the number of times the path on G_k revisits a vertex in G_{k-1}. It is these types of steps that we will suppress in our self-repelling path measure.

For $x > 0$ and $0 \leqq u \leqq 1$, define

$$\Phi_n(x,u) = \sum_{w\in W_n} (\prod_{k=1}^{n} u^{N_k(w)+M_k(w)}) \ x^{L(w)}, \tag{2.2}$$

$$\Theta_n(x,u) = \sum_{w\in W_n^{(1)}} (\prod_{k=1}^{n} u^{N_k(w)+M_k(w)}) \ x^{L_1(w)},$$

$$\Psi_n(x,u) = \sum_{w\in W_n^{(2)}} (\prod_{k=1}^{n} u^{N_k(w)+M_k(w)}) \ x^{L_2(w)}.$$

Let $W_n^{(1)'}$ be the set of the reflection of the paths in $W_n^{(1)}$ with regard to the y-axis. Each $w \in W_n$ can be regarded as a juxtaposition of some elements of $W_n^{(1)} \cup W_n^{(1)'}$ and $W_n^{(2)}$, that is, for some $0 = t_0 < t_1 < \cdots < t_m < L(w)$, $w_1, \cdots, w_m \in W_n^{(1)} \cup W_n^{(1)'}$ and $w_{m+1} \in W_n^{(2)}$,

$$w(t) = w_i(t - t_{i-1}), \quad t_{i-1} \leqq t < t_i, \quad t_i - t_{i-1} = L_1(w_i), \quad i = 1, \cdots, m,$$

$$w(t) = w_{m+1}(t - t_m), \quad t \geq t_m, \quad L(w) - t_m = L_2(w_{m+1}).$$

This decomposition leads to the relation,

$$\Phi_1(x,u) = \Psi_1(x,u)/(1 - 2u\Theta_1(x,u)).$$

In the following, we shall write Φ, Θ and Ψ instead of Φ_1, Θ_1 and Ψ_1. For each u, within the radii of convergence as power series in x, we have the following explicit forms of Φ, Θ and Ψ:

$$\Theta(x,u) = \frac{2ux^2}{(1+ux)(1-2ux)}\{1 + 2(1-u^2)x^2 - 2(1-u)^2ux^3\},$$

$$\Psi(x,u) = \frac{x^2}{(1+ux)(1-2ux)}\{1 + (1+u)x - u(1-u^2)x^2 + 2(1-u)^2u^2x^3\},$$

$$\Phi(x,u) = \frac{x^2\{1 + (1+u)x - u(1-u^2)x^2 + 2(1-u)^2u^2x^3\}}{(1+ux)(1-2ux) - 4u^2x^2\{1 + 2(1-u^2)x^2 - 2u(1-u)^2x^3\}}.$$

Proposition 2.1. *For $n > m$,*

$$\Phi_n(x,u) = \Phi_m(\Phi_{n-m}(x,u),u).$$

Proof. Assume $n > m$ and $w \in W_n$. Note that

$$L(w) = T^m_{L(Q_m w)}(w) = \sum_{i=1}^{L(Q_m w)} \{T^m_i(w) - T^m_{i-1}(w)\} = \sum_{i=1}^{L(Q_m w)} S^m_i(w). \tag{2.3}$$

This together with (2.1) implies that for $k = m+1, \cdots, n$

$$N_k(w) = \sum_{\ell=1}^{L(Q_{k-1}w)} N_k(\ell)(w) = \sum_{i=1}^{L(Q_m w)} \sum_{\ell=T^m_{i-1}(Q_{k-1}w)+1}^{T^m_i(Q_{k-1}w)} N_k(\ell)(w).$$

A similar decomposition holds also for $M_k(w)$. Using these and (2.3), we can rewrite the summands in (2.2) as

$$(\prod_{k=1}^{n} u^{N_k(w)+M_k(w)})\ x^{L(w)} = \prod_{k=1}^{m} u^{N_k(w)+M_k(w)} \cdot \{\prod_{k=m+1}^{n} \prod_{i=1}^{L(Q_m w)} u^{\sum_\ell \{N_k(\ell)(w)+M_k(\ell)(w)\}}\} \cdot \prod_{i=1}^{L(Q_m w)} x^{S^m_i(w)},$$

where $\sum_\ell$ is taken over $\ell = T^m_{i-1}(Q_{k-1}w)+1, \cdots, T^m_i(Q_{k-1}w)$.

For $i = 1, \cdots, L(Q_m w)$, let Δ_i and Δ'_i be two adjacent elements of $\mathcal{T}_m$ determined by

$$w(T^m_{i-1}(w)) \in \Delta_i \cap \Delta'_i,$$

$$w(T^m_i(w)) \in \Delta_i \cap (\Delta'_i)^c.$$

Consider each path segment $w_i = \{\ w(t)\ :\ T^m_{i-1}(w) \leqq t \leqq T^m_i(w)\ \}$. Note that $w_i \subset \Delta_i \cup \Delta'_i$. Since $\Delta_i \cap F_n$ and $\Delta'_i \cap F_n$ are both similar to F'_{n-m}, there is $\tilde{w}_i \in W_{n-m}$ such that $\{\ \tilde{w}_i(t)\ :\ 0 \leqq t \leqq L(\tilde{w}_i)\ \} \cap F'_{n-m}$ and $\{\ \tilde{w}_i(t)\ :\ 0 \leqq t \leqq L(\tilde{w}_i)\ \} \cap (F'_{n-m} - b)$ are similar to $w_i \cap \Delta_i$ and $w_i \cap \Delta'_i$ (or, to their reflections), respectively. In terms of $\tilde{w}_i$'s, the right-hand side of the above expression of the summand can further be rewritten as

$$\begin{aligned}
&\prod_{k=1}^{m} u^{N_k(w)+M_k(w)} \cdot \prod_{i=1}^{L(Q_m w)} (\prod_{k=1}^{n-m} u^{\sum_{\ell=1}^{L(Q_{k-1}\tilde{w}_i)} \{N_k(\ell)(\tilde{w}_i)+M_k(\ell)(\tilde{w}_i)\}})\ x^{L(\tilde{w}_i)} \\
= &\prod_{k=1}^{m} u^{N_k(w)+M_k(w)} \cdot \prod_{i=1}^{L(Q_m w)} (\prod_{k=1}^{n-m} u^{N_k(\tilde{w}_i)+M_k(\tilde{w}_i)})\ x^{L(\tilde{w}_i)}.
\end{aligned}$$

Summing up over W_n, we have

$$\begin{aligned}
&\Phi_n(x,u) \\
= &\sum_{v \in W_m} [\prod_{k=1}^{m} u^{N_k(v)+M_k(v)} \{\Phi_{n-m}(x,u)\}^{L(v)} \\
&\cdot \frac{1}{\{\Phi_{n-m}(x,u)\}^{L(v)}} \sum_{\tilde{w}_1 \in W_{n-m}} \cdots \sum_{\tilde{w}_{L(v)} \in W_{n-m}} \prod_{i=1}^{L(v)} (\prod_{k=1}^{n-m} u^{N_k(\tilde{w}_i)+M_k(\tilde{w}_i)})\ x^{L(\tilde{w}_i)}] \\
= &\sum_{v \in W_m} [\prod_{k=1}^{m} u^{N_k(v)+M_k(v)} \{\Phi_{n-m}(x,u)\}^{L(v)} \cdot \prod_{i=1}^{L(v)} \{\frac{1}{\Phi_{n-m}(x,u)} \sum_{\tilde{w}_i \in W_{n-m}} (\prod_{k=1}^{n-m} u^{N_k(\tilde{w}_i)+M_k(\tilde{w}_i)})\ x^{L(\tilde{w}_i)}\}] \\
= &\sum_{v \in W_m} \prod_{k=1}^{m} u^{N_k(v)+M_k(v)} \{\Phi_{n-m}(x,u)\}^{L(v)} \\
= &\Phi_m(\Phi_{n-m}(x,u),u).
\end{aligned}$$

□

We next define a family of probability measures $\{\tilde{P}^u_n(x)\}$ on C (supported on W_n) by assigning to each $w \in W_n$,

$$\tilde{P}^u_n(x)(w) = (\prod_{k=1}^{n} u^{N_k(w)+M_k(w)})\ x^{L(w)}/\Phi_n(x,u). \tag{2.4}$$

Let $n > m$ and $w \in W_n$. Using the decomposition in the proof of Proposition 2.1 we rewrite (2.4) as

$$\tilde{P}^u_n(x)(w) = \tilde{P}^u_m(\Phi_{n-m}(x,u))(Q_m w) \cdot \prod_{i=1}^{L(Q_m w)} \tilde{P}^u_{n-m}(x)(\tilde{w}_i), \tag{2.5}$$

where $\tilde{w}_i$'s are as in the proof of Proposition 2.1.

The following proposition follows immediately from (2.5).

Proposition 2.2. *If $w \in W_n$ and $m \leqq n$, then $Q_m w \in W_m$. The probability law of $Q_m w$ under $\tilde{P}^u_n(x)$ is $\tilde{P}^u_m(\Phi_{n-m}(x,u))$.*

Let r_u be the radius of convergence for $\Phi(x,u)$ as a power series in x.

Proposition 2.3. *(1) For each $u \in [0,1]$, there is a unique fixed point x_u of the mapping $\Phi(\cdot,u) : (0,r_u) \to (0,\infty)$, that is,*

$$\Phi(x_u,u) = x_u, \qquad x_u > 0.$$

As a function in u, x_u is continuous and strictly decreasing on $[0,1]$.

(2) Let $\lambda_u = \dfrac{\partial \Phi}{\partial x}(x_u,u)$. Then λ_u is continuous in u and $\lambda_u > 2$.

Proof. (1) $\Phi_u(x) = \Phi(x,u)$ is expressed as a power series in x with non-negative coefficients, starting from a quadratic term. It follows that $\Phi_u(0) = \Phi_u'(0) = 0$, $\inf_{x\geq 0} \Phi_u''(x) = \Phi_u''(0) > 0$. Therefore $\Phi_u'(x) - 1$ is increasing in $x \geq 0$, negative at $x = 0$ and diverges to $+\infty$ as $x \to r_u$. Existence and uniqueness of the fixed point follow. The rest of the statement follows from the application of the implicit function theorem to $F(x,u) = \Phi(x,u) - x$.

(2) Since the terms in $\Phi_u(x)$ are degree 2 or higher, and since there are terms of degree strictly higher than 2, we have

$$\Phi_u'(x) > \frac{2}{x}\Phi_u(x), \quad 0 < x < r_u.$$

Therefore $\lambda_u > 2\ \dfrac{\Phi_u(x_u)}{x_u} = 2$. The continuity of x_u in u and the continuity of $\dfrac{\partial \Phi}{\partial x}(x,u)$ in (x,u) imply the continuity of λ_u.

□

In the two extreme cases, we know that $x_0 = \dfrac{\sqrt{5}-1}{2}$, $\lambda_0 = \dfrac{7-\sqrt{5}}{2}$ (see [14], [12]), and $x_1 = \dfrac{1}{4}$, $\lambda_1 = 5$ (see [2], [19]). For $m \leqq n$, let $Q_m \tilde{P}^u_n$ be the image measure of $\tilde{P}^u_n$ induced by Q_m. Combining Proposition 2.2 and Proposition 2.3, we have

Proposition 2.4. *If $w \in W_n$ and $m \leqq n$, then $Q_m w \in W_m$ and*

$$Q_m \tilde{P}^u_n(x_u) = \tilde{P}^u_m(x_u).$$

By virtue of Proposition 2.4 and Kolmogorov's extension theorem, for each $u \in [0,1]$, there is a probability measure P^{*u} on $\Omega = C^{\mathbb{N}} = C \times C \times \cdots$ such that

$$P^{*u}[\ \omega = (w_1, w_2, \cdots)\ :\ Q_m w_n = w_m,\ n \geq m\ ] = 1,$$

and

$$Y_n P^{*u} = \tilde{P}^u_n(x_u),$$

where $Y_n P^{*u}$ denotes the image measure of P^{*u} induced by the natural projection Y_n from Ω to the n-th C in the product. We regard each $Y_n(\omega, t)$ as an F-valued process on $(\Omega, \mathcal{B}, P^{*u})$, where $\mathcal{B}$ is the Borel algebra on Ω. The following properties are used later.

$$Y_n \in W_n, \quad \text{a.s.}$$

$$Q_m Y_n = Y_m, \quad m < n, \quad \text{a.s.}$$

In particular,

$$Y_n(T_i^m(Y_n)) = Y_m(i), \quad m < n, \quad \text{a.s.} \tag{2.6}$$

$$T_i^m(Y_k) = T^n_{T_i^m(Y_n)}(Y_k), \quad m \leqq n \leqq k, \quad \text{a.s.} \tag{2.7}$$

(2.5) and Proposition 2.4 imply the following.

Proposition 2.5. *Assume $n \geq m$ and $N \geq 2^m$. Under the conditional probability $P^{*u}[\ \cdot \ \mid\ Y_m \in W_m,\ L(Y_m) \geq N\ ]$,*

$$\{S_1^m(Y_n),\ S_2^m(Y_n),\ \cdots,\ S_N^m(Y_n)\}$$

are i.i.d. random variables. They are jointly independent of Y_m. The law of $S_1^m(Y_n)$ is equal to that of $S_1^0(Y_{n-m})$ under P^{*u}.

Proposition 2.6. *Fix $m \in \{0, 1, 2, \cdots\}$, $N \geq 2^m$ and $i \in \{0, 1, 2, \cdots, N\}$. Under $P^{*u}[\ \cdot \ \mid\ Y_m \in W_m,\ L(Y_m) \geq N\ ]$, $\{S_i^m(Y_{m+n})\}$, $n = 0, 1, 2, \cdots$ is a supercritical branching process starting at $S_i^m(Y_m) = 1$ and with offspring distribution equal to the law of $S_1^0(Y_1)$ under P^{*u} with the properties:*

(1) The characteristic function of $S_1^0(Y_1)$ is given by

$$E^u[\exp(itS_1^0(Y_1))] = \frac{1}{x_u}\Phi(x_u e^{it}, u), \quad t \in \mathbb{R}.$$

(2) $E^u[S_1^0(Y_1)] = \lambda_u$,

(3) $0 < E^u[(S_1^0(Y_1) - \lambda_u)^2] < +\infty$,

(4) $P^{*u}[S_1^0(Y_1) \geq 2] = 1$.

Proof. (2.7) implies that

$$T_i^m(Y_{m+n+1}) = T^{m+n}_{T_i^m(Y_{m+n})}(Y_{n+m+1}) = \sum_{j=1}^{T_i^m(Y_{m+n})} S_j^{m+n}(Y_{m+n+1}).$$

Hence,

$$\begin{aligned} S_i^m(Y_{m+n+1}) &= \sum_{j=T_{i-1}^m(Y_{m+n})+1}^{T_i^m(Y_{m+n})} S_j^{m+n}(Y_{m+n+1}) \\ &= \sum_{j=1}^{S_i^m(Y_{m+n})} S^{m+n}_{T_{i-1}^m(Y_{m+n})+j}(Y_{m+n+1}). \end{aligned}$$

This combined with Proposition 2.5 implies that $\{S_i^m(Y_{m+n})\}$, $n = 0, 1, 2, \cdots$ is a branching process. (1) is immediate from (2.4), and (2) through (4) are the immediate consequences of (1). Since $\lambda_u > 2$, $\{S_i^m(Y_{m+n})\}$ is a supercritical branching process. □

Proposition 2.6 suggests that we consider F-valued processes with time appropriately scaled. We introduce a time-scale transformation $U_n(\alpha) : C \to C,\ \alpha \in (0,\infty),\ n \in \mathbb{N}$. For $w \in C$, define

$$(U_n(\alpha)w)(t) \stackrel{\text{def}}{=} w(\alpha^n t).$$

Let us denote by P_n^u the image measure of $\tilde{P}_n^u(x_u)$ induced by $U_n(\lambda_u)$. Define

$$X_n = U_n(\lambda_u)Y_n, \quad n = 1, 2, \cdots.$$

The convergence theorem for supercritical branching processes (See [1]) leads to the following proposition.

Proposition 2.7. *Assume $N \geq 2^m$. Under $P^{*u}[\ \cdot \mid Y_m \in W_m,\ L(Y_m) \geq N\ ]$, we have*

(1) For each $i \in \{1, \cdots, N\}$, $S_i^m(X_n)$ converges a.s. and in L^2 as $n \to \infty$ to a random variable S_i^{*m}.

(2) S_i^{*m}, $i = 1, \cdots, N$ are i.i.d. random variables and are jointly independent of Y_m.

(3) S_i^{*m} is equal in law to $\lambda_u^{-m} S_1^{*0}$.

(4) $P^{*u}[S_1^{*0} > 0] = 1$, $E^u[S_1^{*0}] = 1$.
The characteristic function of S_1^{*0}, $\phi_u(t) = E^u[\exp(itS_1^{*0})]$ is the unique solution to

$$\phi_u(\lambda_u t) = \frac{1}{x_u}\Phi(x_u\phi_u(t), u), \quad \phi_u'(0) = 1.$$

We denote by p^{*u} and p_n^u the law of S_1^{*0} and $S_1^0(X_n)$, respectively.

Theorem 2.8. *p^{*u} has a C^∞ density ρ, which satisfies $\rho(x) = 0$ for $x \leqq 0$, and $\rho(x) > 0$ for $x > 0$.*

In general, a C^∞ density exists for the limit distribution of supercritical branching processes such that the number of offspring is almost surely greater than one. In our case the condition is ensured by Proposition 2.6 (2). For a proof see [2], [16].

Let $T_i^{*m} = \sum_{j=1}^{i} S_j^{*m}$.

Theorem 2.9. *For each $u \in [0,1]$, X_n converges uniformly in t a.s. as $n \to \infty$ to a continuous process X.*

Proof. Choose $\omega \in \Omega$ such that $Y_m \in W_m$, $\lim_{n\to\infty} S_i^m(X_n) = S_i^{*m}$ exists and $S_i^{*m} > 0$ for all $m \in \mathbb{N}$ and $i \in \{\ 1, \cdots, T_1^0(Y_m)\ \}$. Let $M = T_1^{*0} + \varepsilon$, where $\varepsilon > 0$ is an arbitrary constant. It suffices to show that $X_n(\omega)$ converges uniformly in $[0, M]$. In fact, if $t > M$, $X_n(t) = a$ for large enough n.

Fix $m \geq 0$. Let $L = T_1^0(Y_m)$. Note that $T_L^m(X_n) = T_1^0(X_n)$ a.s. Letting $n \to \infty$, we have $T_L^{*m} = T_1^{*0}$ a.s.

By the choice of ω, there exists $n_1 = n_1(\omega) \in \mathbb{N}$ such that

$$\max_{0 \leqq i \leqq L} |T_i^m(X_n) - T_i^{*m}| \leqq \min_{0 \leqq i \leqq L} S_i^{*m},$$

and

$$|T_L^m(X_n) - T_L^{*m}| < \varepsilon,$$

for $n \geq n_1$.

For each $t \in [0, M]$, either of the following holds.

(i) $0 \leqq t < T_L^{*m}$.

(ii) $T_L^{*m} \leqq t \leqq T_L^{*m} + \varepsilon$.

In case (i), choose $j \in \{1, \cdots, L\}$, such that $T^{*m}_{j-1} \leqq t < T^{*m}_j$. Then we have $T^m_{j-2}(X_n) \leqq t \leqq T^m_{j+1}(X_n)$, for $n \geq n_1$. Thus

$$|X_n(T^m_j(X_n)) - X_n(t)| \leqq 3 \cdot 2^{-m}.$$

In case (ii), let $j = L$. Since $T^m_{L(X_m)-1}(X_n) \leqq t$,

$$|X_n(T^m_j(X_n)) - X_n(t)| \leqq 2 \cdot 2^{-m}.$$

(2.6) implies that $X_n(T^m_j(X_n)) = Y_m(j)$ for any $n \geq m$ and $j \in \{0, 1, \cdots L(Y_m)\}$. Therefore, if $n, n' \geq n_1$, then for any $t \in [0, M]$,

$$\begin{aligned} & |X_n(t) - X_{n'}(t)| \\ \leqq \; & |X_n(T^m_j(X_n)) - X_n(t)| + |X_{n'}(T^m_j(X_{n'})) - X_{n'}(t)| + |X_n(T^m_j(X_n)) - X_{n'}(T^m_j(X_{n'}))| \\ \leqq \; & 6 \cdot 2^{-m}. \end{aligned}$$

Since m is arbitrary, we have the uniform convergence. □

Corollary 2.10. *Assume $n \geq m$.*

(1) $X(T^{*m}_j) = X_n(T^m_j(X_n)) = Y_m(j) \in G_m$, for all $j \in \{0, 1, \cdots, T^0_1(Y_m)\}$, a.s.

(2) With probability 1, there exist adjacent closed triangles $\Delta, \Delta' \in \mathcal{T}_m$ such that

$$X(T^{*m}_{j-1}) \in \Delta \cap \Delta',$$

$$X(T^{*m}_j) \in \Delta \cap (\Delta')^c,$$

and

$$X(t) \in \Delta \cup \Delta' \quad \text{for } \; T^{*m}_{j-1} \leqq t \leqq T^{*m}_j.$$

(3) $X(t) = a, \quad t \geq T^{*0}_1$, a.s.

Proof. $X_n(T^m_j(X_n)) = Y_m(j)$ for $n \geq m$ and the a.s. uniform convergence of X_n to X imply the statement of (1). The definition of hitting times and (1) imply that $X_n(t) \in \Delta \cup \Delta'$ for $T^m_{j-1}(X_n) \leqq t \leqq T^m_j(X_n)$ a.s. Letting $n \to \infty$, we have (2). (3) follows from $X_n(t) = a$, $t \geq T^0_1(X_n)$ a.s. □

This result implies that the limit process for $0 < u < 1$ maintains the self-repelling property of the original walk when observed at the 2^{-n}-scale for each $n \in \mathbb{N}$.

3 Continuity of the limit processes in the self-repelling parameter.

In this section, we show that the process is 'continuous' in u. Denote by P^u and P^u_n the laws of X and X_n under P^{*u}, respectively. P^u and P^u_n are measures on C. We will show $P^u \to P^{u_0}$ weakly as $u \to u_0$.

We start with the 'continuity' of p^{*u} in u. Since p^u_n and p^{*u} have supports on $[0, \infty)$ as stated in Theorem 2.8, their Laplace transforms,

$$G^u_n(s) \stackrel{\text{def}}{=} \int_0^\infty \exp(-s\eta)\; p^u_n(d\eta) = \frac{1}{x_u}\; \Phi_n(x_u \exp(-\lambda_u^{-n} s), u),$$

and

$$G^u(s) \stackrel{\text{def}}{=} \int_0^\infty \exp(-s\eta)\; p^{*u}(d\eta).$$

are holomorphic in $\{\; s \in \mathbb{C} \;:\; \Re(s) > 0 \;\}$.

Let

(3.1) $$g^u_n(s) \stackrel{\text{def}}{=} -\log\{\frac{1}{x_u}\Phi_n(x_u \exp(-\lambda_u^{-n} s), u)\}, \; s \in \mathbb{C},$$

and

$$H^u(q) \stackrel{\rm def}{=} \frac{1}{x_u}\Phi(x_u q, u).$$

Proposition 2.1 with $m = n-1$ and $x = x_u$ implies

$$\exp(-g_n^u(s)) = H^u(\exp(-g_{n-1}^u(\lambda_u^{-1}s)))$$

and

$$g_n^u(s) = g_1^u(\lambda_u g_{n-1}^u(\lambda_u^{-1}s)). \tag{3.2}$$

Proposition 3.1. *There exist positive constants C_∞ and M_∞ such that for any $u \in [0,1]$ and any $n \in \mathbb{N}$, $g_n^u(s)$ is holomorphic on $|s| \leqq C_\infty$ and satisfies*

$$|g_n^u(s) - s| \leqq M_\infty |s|^2, \quad |s| \leqq C_\infty \tag{3.3}$$

Proof. The implicit function theorem implies that for each $u \in [0,1]$, there exists a positive solution $x = a_u$ to

$$\Phi(x,u) = 1$$

and that a_u is a continuous function of u. Note that $\Phi(x_u,u) = x_u \leqq x_0 < 1$. Hence $C_1 \stackrel{\rm def}{=} \min_{u\in[0,1]} \frac{a_u}{x_u} > 1$. Since $|\frac{1}{x_u}\Phi(x_u e^{-s},u) - 1|$ is finite and continuous on the compact set $\{(u,s) \in [0,1]\times\mathbb{C} \;:\; |s| \leqq \log C_1\}$, it is uniformly continuous. It follows from this and $\Phi(x_u,u) = x_u$ that there exists $C_2 > 0$ such that $|\frac{1}{x_u}\Phi(x_u e^{-s},u) - 1| < \frac{1}{2}$ for $\{(u,s) \in [0,1]\times\mathbb{C} \;:\; |s| \leqq C_2\}$. Thus (3.1) implies that $g_1^u(s)$ is holomorphic in $\{s \in \mathbb{C} \;: |s| \leqq C_2/\lambda_u\}$. This combined with $g_1^u(0) = 0$ and $(g_1^u)'(0) = 0$, implies that there exist positive constants C and M such that for any $u \in [0,1]$, $g_1^u(s)$ is holomorphic on $\{s \in \mathbb{C} \;:\; |s| < C\}$ and satisfies

$$|g_1^u(s) - s| \leqq M|s|^2, \quad |s| \leqq C \tag{3.4}$$

Fix $\varepsilon > 0$. Let $\lambda_- = \inf_{u\in[0,1]} \lambda_u$, and

$$M_\infty = \frac{M(1+\varepsilon)^2}{1-\frac{1}{\lambda_-}}, \quad C_\infty = \frac{\varepsilon\lambda_-}{M_\infty} \wedge \tilde{C}, \tag{3.5}$$

where $\tilde{C}$ is a positive constant defined by

$$\tilde{C}\left(1 + \frac{\tilde{C}M_\infty}{\lambda_-}\right) = C. \tag{3.6}$$

Note that $C_\infty \leqq \tilde{C} < C$.

Define a sequence M_n, $n = 1,2,3,\cdots$, by

$$M_n = M_\infty\left(1 - \frac{1}{\lambda_-^n}\right) - \frac{M\varepsilon(2+\varepsilon)}{\lambda_-^{n-1}}.$$

It is straightforward to see that

$$\begin{aligned}
M_n &< M_\infty, \quad n = 1,2,3,\cdots, && (3.7)\\
M_1 &= M, && (3.8)\\
M_{n+1} &= M_\infty\left(1 - \frac{1}{\lambda_-}\right) + \frac{1}{\lambda_-}M_n \\
&= M(1+\varepsilon)^2 + \frac{1}{\lambda_-}M_n, \quad n = 1,2,3,\cdots. && (3.9)
\end{aligned}$$

We prove by induction in n that $g_n^u(s)$ is holomorphic on $|s| \leqq C_\infty$ and satisfies

$$|g_n^u(s) - s| \leqq M_n|s|^2, \quad |s| \leqq C_\infty\,. \tag{3.10}$$

Then (3.7) and (3.10) imply Proposition 3.1.

$C_\infty < C$, (3.4) and (3.8) imply that $g_1^u(s)$ is holomorphic on $|s| \leqq C_\infty$ and (3.10) holds for $n = 1$.

Assume that for $n = k$, $g_k^u(s)$ is holomorphic on $|s| \leqq C_\infty$ and (3.10) holds. Note that $\lambda_- > 1$ implies that if $|s| \leqq C_\infty$ then $|s/\lambda_u| < C_\infty$. Hence by induction hypothesis, $g_k^u(s/\lambda_u)$ is holomorphic and satisfies

$$\lambda_u |g_k^u(s/\lambda_u) - s/\lambda_u| \leqq \frac{M_k}{\lambda_u}|s|^2. \tag{3.11}$$

This and (3.7) and (3.6) further imply, for $|s| \leqq C_\infty$,

$$\lambda_u |g_k^u(s/\lambda_u)| \leqq |s|\left(1 + \frac{M_k}{\lambda_u}|s|\right) \leqq C_\infty\left(1 + \frac{M_k}{\lambda_-}C_\infty\right) \leqq C.$$

Therefore $g_{k+1}^u(s) = g_1^u(\lambda_u g_k^u(s/\lambda_u))$ is holomorphic on $|s| \leqq C_\infty$ and

$$\begin{aligned} &|g_1^u(\lambda_u g_k^u(s/\lambda_u)) - \lambda_u g_k^u(s/\lambda_u)| \\ &\leqq M|\lambda_u g_k^u(s/\lambda_u)|^2 \leqq M\left(|s| + \frac{M_k}{\lambda_u}|s|^2\right)^2 \leqq M\left(1 + \frac{M_k C_\infty}{\lambda_-}\right)^2 |s|^2 \\ &\leqq M(1+\varepsilon)^2|s|^2, \end{aligned} \tag{3.12}$$

where we also used (3.7) and $C_\infty \leqq \dfrac{\varepsilon\lambda_-}{M_\infty}$ in the last line.

Using (3.11) and (3.12) we finally have

$$\begin{aligned} &|g_{k+1}^u(s) - s| \leqq |g_1^u(\lambda_u g_k^u(s/\lambda_u)) - \lambda_u g_k^u(s/\lambda_u)| + \lambda_u|g_k^u(s/\lambda_u) - s/\lambda_u| \\ &\leqq \left(\frac{M_k}{\lambda_u} + M(1+\varepsilon)^2\right)|s|^2 \leqq \left(\frac{M_k}{\lambda_-} + M(1+\varepsilon)^2\right)|s|^2 = M_{k+1}|s|^2, \quad |s| \leqq C_\infty, \end{aligned}$$

which implies (3.10) for $n = k+1$. By induction, we have (3.10) for all n. □

Proposition 3.2. *There is a positive constant C_∞ such that for each $u \in [0,1]$, $g_n^u(s)$ converges uniformly on any compact subset of $\{ s \in \mathbb{C} \ : \ |s| < C_\infty\}$ to a holomorphic function $g^u(s)$ as $n \to \infty$. $g^u(s)$ satisfies the following functional relation:*

$$g^u(s) = g_1^u(\lambda_u g^u(\lambda_u^{-1}s)), \tag{3.13}$$

and

$$\exp(-g^u(s)) = H^u(\exp(-g^u(\lambda_u^{-1}s))). \tag{3.14}$$

Proof. Fix $u \in [0,1]$. (3.3) implies that the family of functions, $\{g_n^u(s) \ : \ n = 1,2,\cdots\}$ is uniformly bounded. This combined with $g_n^u(0) = 0$ implies that $\{g_n^u(s) \ : \ n = 1,2,\cdots\}$ forms a normal family on $\{s \in \mathbb{C} : |s| < C_\infty\}$. Therefore for any subsequence of $\{g_n^u(s)\}$, there exists a sub-subsequence that converges to a holomorphic function g^* uniformly on any compact subset of $\{s \in \mathbb{C} : |s| < C_\infty\}$. Note that

$$\exp(-g_n^u(s)) = G_n^u(s)$$

on $\{s \in \mathbb{C} : |s| < C_\infty, \ \Re(s) > 0\}$. Proposition 2.7 implies that p_n^u converges weakly to p^{*u} as $n \to \infty$, which further implies $G_n^u(s) \longrightarrow G^u(s)$ as $n \to \infty$. Thus

$$g^*(s) = -\log G^u(s)$$

on $\{s \in \mathbb{C} : |s| < C_\infty, \ \Re(s) > 0\}$. This combined with the principle of analytic continuation implies that the limit g^* is independent of the subsequences. Therefore $\{g_n^u\}$ converges uniformly on any compact subsets of $\{s \in \mathbb{C} : |s| < C_\infty\}$ to a holomorphic function g^u satisfying

$$\exp(-g^u(s)) = G^u(s), \tag{3.15}$$

on $\{s \in \mathbb{C} : |s| < C_\infty, \ \Re(s) > 0 \}$. Letting $n \to \infty$ in (3.2), we have (3.13). Rewriting (3.13), we also have (3.14).

□

Let

$$F_0(s) = \exp(-g^u(s)),$$

$$F_n(s) = (H^u)^n(\exp(-g^u(\lambda_u^{-n}s))),\ n = 1, 2, \cdots,$$

and $A_0 = \{s \in \mathbb{C} \ : \ |s| < C_\infty \}$, $A_n = \{s \in \mathbb{C} \ : \ \lambda_u^{n-1}C_\infty \leqq |s| < \lambda_u^n C_\infty \}$, for $n = 1, 2, \cdots$. Define $\tilde{G}^u : \mathbb{C} \longrightarrow \mathbb{C}$ by

$$\tilde{G}^u(s) = F_n(s), \text{ on } A_n, \quad n = 0, 1, 2, \cdots. \tag{3.16}$$

Since $H(q)$ is a rational function and (3.14) implies $F_n(s) = F_{n-1}(s)$ on $\bigcup_{k=0}^{n-1} A_k$, $\tilde{G}^u(s)$ is meromorphic on $\mathbb{C}$. This combined with (3.15) implies

$$\tilde{G}^u(s) = G^u(s), \text{ on } \{ s \in \mathbb{C} \ : \ \Re(s) > 0 \}. \tag{3.17}$$

Proposition 3.3. *For any $u_0 \in [0, 1]$, p^{*u} converges weakly to p^{*u_0} as $u \to u_0$.*

Proof. Letting $n \to \infty$ in (3.3), we see that

$$|g^u(s) - s| \leqq M_\infty |s|^2, \quad |s| < C_\infty.$$

This implies $\{ g^u : u \in [0, 1] \}$ forms a normal family on $\{ s \in \mathbb{C} \ : \ |s| < C_\infty \}$. Let $\{u_n\}$ be a sequence in $[0, 1]$ that converges to u_0. Then $\{ g^{u_n} : n = 1, 2, \cdots \}$ has a subsequence that converges to a holomorphic function h uniformly on any compact subset of $\{ s \in \mathbb{C} \ : \ |s| < C_\infty \}$. h satisfies the functional relation,

$$h(s) = g_1^{u_0}(\lambda_{u_0} h(\lambda_{u_0}^{-1} s)).$$

$$h(0) = 0, \quad h'(0) = 1.$$

This equation has a unique solution g^{u_0}. Thus $g^{u_n} \longrightarrow g^{u_0}$ uniformly on any compact subset of $\{ s \in \mathbb{C} \ : |s| < C_\infty \}$ as $n \to \infty$. This combined with (3.16) implies that $\tilde{G}^{u_n}(s) \longrightarrow \tilde{G}^{u_0}(s)$ for every $s \in \mathbb{C}$ with $\Re(s) > 0$. We have, from (3.17),

$$\lim_{n \to \infty} G^{u_n}(s) = G^{u_0}(s), \quad \text{on } \{ s \in \mathbb{C} \ : \ \Re(s) > 0 \}.$$

Since a Laplace transform determines a measure, we see that p^{*u_n} converges weakly to p^{*u_0} as $n \to \infty$.

□

Now we go on to the 'continuity' of P^u in u, where P^u is defined at the beginning of this section.

Proposition 3.4. *The family of measures $\{P^u\}$, $0 \leqq u \leqq 1$ is tight.*

Proof. Since $\{ w \in C \ : \ w(0) = O \}$ is a closed subset of C, we have

$$P^u[\ w(0) = O\ ] \geq \limsup_{n \to \infty} P_n^u[\ w(0) = O\ ] = 1.$$

We will show that for any $\varepsilon > 0$ and $u_0 \in [0, 1]$, there exist positive numbers α and δ such that

$$P^u[\ \sup_{|s-t|<\delta} |w(s) - w(t)| > \varepsilon\ ] \leqq \varepsilon, \quad \text{for any } u \text{ with } |u - u_0| < \alpha.$$

For an arbitrarily given ε, choose $k \in \mathbb{Z}$ satisfying

$$2 \cdot 2^{-k} < \varepsilon.$$

Choose N large enough so that

$$\tilde{P}_k^u[L(w) > N] < \frac{\varepsilon}{2}, \quad \text{for all } u \in [0, 1].$$

Let

$$V = \{w \in W_k \ : \ L(w) \leqq N\},$$

and

$$D = \{w \in C \ : \text{ There exist } s,t \geq 0 \text{ with } |s-t| < \delta \text{ and } \Delta_1, \Delta_2 \in \mathcal{T}_k \\ \text{with } \Delta_1 \cap \Delta_2 = \emptyset \text{ such that } w(s) \in int(\Delta_1), w(t) \in int(\Delta_2)\ \},$$

where $int(\Delta_i) = \Delta_i \cap G_k^c$, $i = 1, 2$. D is an open subset of C. Note that Theorem 2.9 implies that $P_n^u \longrightarrow P^u$ weakly as $n \to \infty$. The choice of k and the continuity of w imply that if $\sup_{|s-t|<\delta} |w(s) - w(t)| > \varepsilon$, then $w \in D$.

We have,

$$\begin{aligned}
&P^u[\sup_{|s-t|<\delta} |w(s) - w(t)| > \varepsilon\ ] \\
&\leqq\ P^u[D] \\
&\leqq\ \liminf_{n\to\infty} P_n^u[D] \\
&\leqq\ \liminf_{n\to\infty} P_n^u[\ S_i^k < \delta \text{ for some } i = 1, \ldots, L(Q_k w)\ ] \\
&\leqq\ \liminf_{n\to\infty}\{\sum_{v\in V}\sum_{i=1}^{L(v)} P_n^u[\ S_i^k < \delta \mid Q_k w = v\ ] \cdot P_n^u[\ Q_k w = v\ ] + P_n^u[\ Q_k w \in W_k \setminus V\ ]\} \\
&<\ N \sum_{v\in V} \lim_{n\to\infty} p_n^u[\ s : s < \lambda_u^k \delta\ ]\ \tilde{P}_k^u[\ v\ ] + \frac{\varepsilon}{2} \\
&=\ N \sum_{v\in V} p^{*u}[\ s : s < \lambda_u^k \delta\ ] \tilde{P}_k^u[\ v\ ] + \frac{\varepsilon}{2} \\
&\leqq\ N\ p^{*u}[\ s : s < \lambda_u^k \delta\ ] + \frac{\varepsilon}{2},
\end{aligned}$$

where we used Proposition 2.4, Proposition 2.5 and Theorem 2.8.

The continuity of p^{*u} in u proved in Proposition 3.3 combined with $p^{*u}[\{0\}] = 0$ implied by Theorem 2.8 shows that we can choose a $\delta > 0$ such that

$$p^{*u}[\ s : s < \lambda_u^k \delta\ ] \leqq p^{*u}[\ s : s < (\max_{u\in[0,1]} \lambda_u)^k \delta\ ] \leqq \frac{\varepsilon}{2N}, \quad \text{for all } u \in [0,1].$$

Then we have

$$P^u[\sup_{|s-t|<\delta} |w(s) - w(t)| > \varepsilon\ ] < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \text{for all} \ \ u \in [0,1].$$

This completes the proof.

□

Now we will show the convergence of the finite dimensional distributions. Fix an arbitrary $m \in \mathbb{Z}$ and $0 < t_1 < \ldots < t_m$. For $w \in C$, let us define

$$h(w, x) = e^{ix_1 \cdot w(t_1) + \ldots + ix_m \cdot w(t_m)},\ x = (x_1, \ldots, x_m) \in (\mathbb{R}^2)^m,$$

where $x \cdot y$ denotes the inner product in $\mathbb{R}^2$. For a probability measure Q on C, define $F(Q) : (\mathbb{R}^2)^m \to \mathbb{C}$ by,

$$F(Q)(x) \stackrel{\text{def}}{=} E^Q[h(\cdot, x)].$$

Proposition 3.5. *For each $x \in (\mathbb{R}^2)^m$, $F(P^u) = F(P^u)(x)$ is continuous in $u \in [0,1]$.*

Proof. Fix $x \in (\mathbb{R}^2)^m$. Let ε be an arbitrary positive number. Define $f : (\mathbb{R}^2)^m \longrightarrow \mathbb{R}$, by $f(y) = \exp(i\sum_{j=1}^m x_j \cdot y_j)$, $y = (y_1, \cdots, y_m) \in (\mathbb{R}^2)^m$. Since f is uniformly continuous, we can choose a positive integer k such that

$$(3.18) \qquad |f(y) - f(z)| < \varepsilon, \quad \text{for any} \ \ y, z \in (\mathbb{R}^2)^m, \quad \text{with } |y_j - z_j| < 2^{-k},\ j = 1, \cdots, m,$$

where $y = (y_1, \cdots, y_m)$, $z = (z_1, \cdots, z_m)$. Furthermore, we can choose a positive integer N such that

$$\tilde{P}_k^u[L(w) > N] < \varepsilon, \quad \text{for all} \ \ u \in [0,1].$$

Let $V = \{ \ w \in W_k \ : \ L(w) \leqq N \ \}$.

For $n \geq k$,

$$
\begin{aligned}
& F(P^u) \\
&= E^{P^{*u}}[\ h(X,x) \ ] \\
&= \sum_{v \in V} E^{P^{*u}}[\ h(X,x) \ \mid X_k = v \ ] \ P^{*u}[\ X_k = v \ ] \\
&\quad + E^{P^{*u}}[\ h(X,x) \mid X_k \in W_k \setminus V \ ] \ P^{*u}[\ X_k \in W_k \setminus V \ ].
\end{aligned}
$$

The first term on the right-hand side is further decomposed as

$$
\begin{aligned}
& \sum_{v \in V} E^{P^{*u}}[\ h(X,x) \ \mid X_k = v \ ] \ P^{*u}[\ X_k = v \ ] \\
&= \sum_{v \in V} \sum_{\{r_i\}} E^{P^{*u}}[\ h(X,x) \mid X_k = v, \ T^{*k}_{r_i} \leqq t_i < T^{*k}_{r_i+1}, \ i = 1, \ldots, m \ ] \\
&\quad \times P^{*u}[\ T^{*k}_{r_i} \leqq t_i < T^{*k}_{r_i+1}, \ i = 1, \ldots, m \ \mid X_k = v \ ] \ P^{*u}[\ X_k = v \ ],
\end{aligned}
$$

where $\sum_{\{r_i\}}$ is taken over $(r_1, \cdots, r_m) \in \{1, 2, \ldots, L(v)\}^m$ with $r_1 \leqq r_2 \leqq \ldots \leqq r_m$. Using $|h(w,x)| \leqq 1$ and the definition of N and V,

$$
\begin{aligned}
& |E^{P^{*u}}[\ h(X,x) \mid X_k \in W_k \setminus V \ ] \ P^u[\ Q_k w \in W_k \setminus V \ ]| \\
& \leqq \ P^u[\ Q_k w \in W_k \setminus V \ ] < \varepsilon.
\end{aligned}
$$

Denote for simplicity,

$$
E(u) = E^{P^{*u}}[\ h(X,x) \mid X_k = v, \ T^{*k}_{r_i} \leqq t_i < T^{*k}_{r_i+1}, \ i = 1, \cdots, m \ ],
$$

$$
\overline{P}^u_1 = P^{*u}[\ T^{*k}_{r_i} \leqq t_i < T^{*k}_{r_i+1}, \ i = 1, \cdots, m \mid X_k = v \ ],
$$

$$
\overline{P}^u_2 = P^{*u}[\ X_k = v \ ].
$$

Fix $u_0 \in [0,1]$ arbitrary. For any $u \in [0,1]$,

$$
\begin{aligned}
& |F(P^u) - F(P^{u_0})| \\
& < \sum_{v \in V} \sum_{\{r_i\}} | \ E(u) - E(u_0) \ | \ \overline{P}^u_1 \ \overline{P}^u_2 + \sum_{v \in V} \sum_{\{r_i\}} |E(u_0)| \ | \ \overline{P}^u_1 - \overline{P}^{u_0}_1 \ | \ \overline{P}^u_2 \\
& + \sum_{v \in V} \sum_{\{r_i\}} |E(u_0)| \ \overline{P}^{u_0}_1 \ | \ \overline{P}^u_2 - \overline{P}^{u_0}_2 | + 2\varepsilon.
\end{aligned}
$$

Corollary 2.10 implies that if $X_k = v$ then $X(T^{*k}_{r_i}) = v(r_i)$ a.s. and that if $T^{*k}_{r_i} \leqq t_i < T^{*k}_{r_i+1}$, then $X(t_i)$ is almost surely either in $\Delta \in \mathcal{T}_k$ such that $v(r_i), \ v(r_i+1) \in \Delta$ or in its neighboring elements of $\mathcal{T}_k$ adjacent at $v(r_i)$. This means $|X(t_i) - v(r_i)| \leqq 2^{-k}$ a.s.

Therefore,

$$
\begin{aligned}
& |E(u) - E(u_0)| \\
&= |E^{P^u}[\ f(w(t_1), \cdots, w(t_m)) - f(v(r_1), \cdots, v(r_m)) \mid X_k = v, \ T^{*k}_{r_i} \leqq t_i < T^{*k}_{r_i+1}, \ i = 1, \cdots, m \ ] \\
&+ \ f(v(r_1), \cdots, v(r_m)) \\
&- \ E^{P^{u_0}}[\ f(w(t_1), \cdots, w(t_m)) - f(v(r_1), \cdots, v(r_m)) \mid X_k = v, \ T^{*k}_{r_i} \leqq t_i < T^{*k}_{r_i+1}, \ i = 1, \cdots, m \ ] \\
&- \ f(v(r_1), \cdots, v(r_m))| \\
&< \ 2\varepsilon,
\end{aligned}
$$

where we have used (3.18) for the last inequality.

Theorem 2.8 and Proposition 3.3 imply that $\overline{P}^u_1$ is continuous in u, thus there exists $\delta_1 > 0$ such that

$$
|\overline{P}^u_1 - \overline{P}^{u_0}_1| < \varepsilon, \quad \text{for any } u \ \text{ with } \ |u - u_0| < \delta_1.
$$

If we note that $P^{*u}[X_k = v] = \tilde{P}_k^u[w = v]$ for $v \in W_k$, we see that $P^{*u}[X_k = v]$ is continuous in u. Thus there exists $\delta_2 > 0$ such that

$$|\overline{P}_2^u - \overline{P}_2^{u_0}| < \varepsilon, \quad \text{for any } u \text{ with } |u - u_0| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then

$$|F(P^u) - F(P^{u_0})| < 6\varepsilon \quad \text{for any } u \text{ with } |u - u_0| < \delta.$$

This completes the proof. □

Proposition 3.4 combined with Proposition 3.5 leads to the following theorem.

Theorem 3.6. *For any $u_0 \in [0, 1]$, P^u converges to P^{u_0} weakly as $u \to u_0$.*

4 Path properties.

Let

$$\gamma = \gamma_u = \frac{\log 2}{\log \lambda_u}, \quad \beta = \frac{1 - \gamma}{\gamma}.$$

Large deviation estimates for the supercritical branching process allow us to state the following Lemma.

Lemma 4.1 ([4]). *There exists a multiplicatively periodic function H such that*

$$-\log P^{*u}(S_1^{*0} < x) = x^{-1/\beta} H(x) + o(x^{-1/\beta}).$$

Using this result we have

Proposition 4.2. *There exist positive constants $C_{2,1} - C_{2,4}$ and K such that*

$$C_{2,1} \exp(-C_{2,2}(\delta t^{-\gamma})^{\frac{1}{1-\gamma}}) \leqq P^{*u}[|X(t)| \geq \delta] \tag{4.1}$$

for all $t \geq 0$ and $0 < \delta < \frac{1}{4}$ with $\delta t^{-\gamma} \geq K$, and

$$P^{*u}[|X(t)| \geq \delta] \leqq P^{*u}[\sup_{0<s\leqq t} |X(s)| \geq \delta] \leqq C_{2,3} \exp(-C_{2,4}(\delta t^{-\gamma})^{\frac{1}{1-\gamma}}), \tag{4.2}$$

for all $t \geq 0$ and $0 < \delta < 1$.

Proof. Consider $\omega \in \Omega$ such that $\lim_{n\to\infty} S_i^m(X_n)(\omega) = S_i^{*m}(\omega)$ exists and $S_i^{*m}(\omega) > 0$ for all m and i.

Lemma 4.1 implies that there exist positive constants $C_{1,1} - C_{1,4}$ such that

$$C_{1,1} \exp(-C_{1,2} x^{-\frac{\gamma}{1-\gamma}}) \leqq P^{*u}[\, S_1^{*0} < x \,] \leqq C_{1,3} \exp(-C_{1,4} x^{-\frac{\gamma}{1-\gamma}}), \quad x \geq 0\,. \tag{4.3}$$

For the lower bound, choose $n \in \{2, 3, \cdots\}$ such that $2^{-n-1} < \delta \leqq 2^{-n}$. Note that Corollary 2.10 implies that if $T_1^{*n-1} < t$ and $S_j^{*n} > t$ for $j = S_1^{n-1}(Y_n) + 1$ then $|X(t)| \geq \delta$ a.s. In terms of branching processes, since T_1^{*n-1} and S_j^{*n} with $j = S_1^{n-1}(Y_n) + 1$ are related to the limit of the numbers of offsprings coming from different children, they are independent. (In other words, S_j^{*n} is independent of $\sigma[S_1^{n-1}(Y_{n-1+r}) : r = 0, 1, \cdots]$.) These combined with (4.3) imply

$$\begin{aligned}
&P^{*u}[|X(t)| \geq \delta] \\
&\quad \geq \; P^{*u}[\, T_1^{*n-1} < t, \; S_j^{*n} > t \,] \\
&\quad = \; P^{*u}[\, T_1^{*n-1} < t \,] \, P^{*u}[\, T_1^{*n} > t \,] \\
&\quad \geq \; C_{1,1} \exp(-C_{1,2}(\lambda_u^{n-1} t)^{-\frac{\gamma}{1-\gamma}}) \{1 - C_{1,3} \exp(-C_{1,4}(\lambda_u^n t)^{-\frac{\gamma}{1-\gamma}}\} \\
&\quad \geq \; C_{1,1} \exp(-4C_{1,2}(\delta t^{-\gamma})^{\frac{1}{1-\gamma}}) \{1 - C_{1,3} \exp(-C_{1,4}(\delta t^{-\gamma})^{\frac{1}{1-\gamma}}\}).
\end{aligned}$$

Choose $K > 0$ large enough so that the last factor exceeds $\frac{1}{2}$ for $\delta t^{-\gamma} \geq K$.

For the upper bound, choose $n \in \mathbb{N}$ such that $2^{-n} < \delta \leqq 2^{-n+1}$. Let $\Delta, \Delta' \in \mathcal{T}_n$ be the adjacent triangles such that $O \in \Delta \cap \Delta'$ and $X(T_1^{*n}) \in \Delta \cap (\Delta')^c$.

Corollary 2.10 implies that for $0 \leqq s \leqq T_1^{*n}$, $X(s) \in \Delta \cup \Delta'$, thus $|X(s)| \leqq 2^{-n} < \delta$. It follows that if $\sup_{0<s\leqq t} |X(s)| \geq \delta$, then $T_1^{*n} < t$. Combining this with Lemma 4.1, we have

$$\begin{aligned} & P^{*u}[\ |X(t)| \geq \delta\ ] \\ & \leqq \ P^{*u}[\ T_1^{*n} < t\ ] = P^{*u}[\ S_1^{*0} < \lambda_u^n t\ ] \\ & \leqq \ C_{1,3} \exp(-C_{1,4} (\lambda_u^n t)^{-\frac{\gamma}{1-\gamma}}) \\ & \leqq \ C_{1,3} \exp(-C_{1,4} (\frac{\delta t^{-\gamma}}{2})^{\frac{1}{1-\gamma}}). \end{aligned}$$

This completes the proof. □

For a sharper result we could obtain large deviation estimates using the approach of [3].

Integrating (4.1) and (4.2), we see that for each $p > 0$ there exist positive constants $C_{3,1}(p)$, $C_{3,2}(p)$ and $\tau(p)$ such that

$$C_{3,1}(p) t^{\gamma p} \leqq E^u[|X(t)|^p] \leqq C_{3,2}(p) t^{\gamma p},$$

for any t with $t < \tau(p)$. Thus we have

Theorem 4.3. *For each $p > 0$, there are positive constants $C_{3,1}(p)$ and $C_{3,2}(p)$ such that*

$$C_{3,1}(p) \leqq \liminf_{t\downarrow 0} \frac{E^u[|X(t)|^p]}{t^{\gamma p}} \leqq \limsup_{t\downarrow 0} \frac{E^u[|X(t)|^p]}{t^{\gamma p}} \leqq C_{3,2}(p).$$

We can conclude with a law of the iterated logarithm for our self-repelling process.

Theorem 4.4. *There exists a positive constant c such that*

$$c \leqq \limsup_{t\downarrow 0} \frac{|X(t)|}{\psi(t)} \leqq 1, \quad P^{*u}\ \text{- a.s.},$$

where $\psi(t) = C_{2,4}^{\gamma-1} t^{\gamma} (\log\log \frac{1}{t})^{1-\gamma}$.

Proof. The upper bound is straightforward to prove using (4.2) with a standard Borel-Cantelli argument.

The lower bound is more difficult and the standard approach applied to Brownian motion cannot be used as it relies on the Markov property of the process. In our setting we do have a distributional self-similarity property for our path which we can exploit.

In order to prove this result we consider the sequence of stopping times $\{T_1^{*n} \mid n \geq 0\}$. Thus $|X_{T_1^{*n}}| = 2^{-n}$ under P^{*u}. We can describe the sequence of times via the limiting random variable in the supercritical branching process defined in Proposition 2.7. Note that for $k < m$,

$$S_1^{*k} = \sum_{j=1}^{S_1^k(Y_m)} S_j^{*(m)}\ ,\ a.s., \tag{4.4}$$

where the summands are i.i.d. and equal in law to $\lambda_i^{-m} S_1^{*0}$.

The behaviour of the asymptotics of a sequence of random variables satisfying this type of equation is discussed in [10, 11] in the context of random recursive fractals. Here we have the somewhat easier task of proving an almost sure lower bound on the oscillation in S_1^{*0}.

Our result will follow from the following Lemma.

Lemma 4.5. *There exist positive constants c, N_0 such that if $m_k = kN$ for $N > N_0$, then*

$$P^{*u}(\lambda_u^{m_k} S_1^{*m_k} \leqq c(\log(m_k))^{-\beta}\ \ i.o.) = 1.$$

Proof. Let $A_m = \{\lambda_u^m S_1^{*m} \leqq \delta(\log m)^{-\beta}\}$ and write A_m^c for the complementary event. Using the fact that there is a Markov structure (4.4) inherited from the branching process in the sequence of random variables $\lambda_u^m S_1^{*m}$ we have

$$P^{*u}(A_m \mid A_{m-1}^c, \dots, A_2^c) = P^{*u}(A_m \mid A_{m-1}^c). \tag{4.5}$$

Then a straightforward extension of the second Borel–Cantelli Lemma shows that we will have the claim of the Lemma, $P^{*u}(\limsup_{k\to\infty} A_k) = 1$, if

$$\sum_{m=1}^{\infty} P^{*u}(A_m | A_{m-1}^c) = \infty. \tag{4.6}$$

Note that as

$$\sum_{m=1}^{\infty} P^{*u}(A_m \mid A_{m-1}^c) = \sum_{m=1}^{\infty} \frac{P^{*u}(A_m \cap A_{m-1}^c)}{P^{*u}(A_{m-1}^c)} \geq \sum_{m=1}^{\infty} P^{*u}(A_m \cap A_{m-1}^c)$$

it is enough to establish

$$\sum_{m=1}^{\infty} P^{*u}(A_m \cap A_{m-1}^c) = \infty \tag{4.7}$$

to obtain (4.6) and prove the claim.

At this stage we consider A_{m_k}, where $m_k = kN$ for some integer N. Using [10] Lemma 4.2 we let $x_k = b^{-\beta}\lambda_u^{-m_k}(\log(m_k))^{-\beta}$ $(b > 0)$ and hence

$$\begin{aligned}
&P^{*u}(S_1^{*m_k} \leqq x_k,\, S_1^{*m_{k-1}} > x_{k-1}) \\
&\quad = P^{*u}(S_1^{*m_k} \leqq x_k,\, \sum_{i=1}^{S_1^{m_{k-1}}(Y_{m_k})} S_i^{*m_k} > x_{k-1}) \\
&\quad = \int_0^{x_k} P^{*u}\left(\sum_{i=2}^{S_1^{m_{k-1}}(Y_{m_k})} S_1^{*m_{k-1}} + y > x_{k-1}\right) P^{*u}(S_1^{*m_k} \in dy) \\
&\quad \geq P^{*u}(S_1^{*m_k} \in [c_1 x_{k-1},\, x_k]) P^{*u}(\sum_{i=2}^{S_1^{m_{k-1}}(Y_{m_k})} S_1^{*m_k} > (1-c_1)x_{k-1}),
\end{aligned}$$

for some constant $0 < c_1 < 1$. Observing that, as x_k is decreasing in k, the second term in the product will be bounded below by a constant c_2. If we now set $c_3 = (c_1\lambda^{m_k - m_{k-1}})^{-1/\beta}$ and apply the tail estimates in Lemma 4.1, then

$$\begin{aligned}
&P^{*u}(S_1^{*m_k} \leqq x_k,\, S_1^{*m_{k-1}} > x_{k-1}) \\
&\quad \geq c_2 \exp(-b\log(m_k) H((b\log(m_k))^{-\beta})) \\
&\quad\quad \times \left(1 - \exp\left(-c_3 b\log(m_{k-1}) H((c_3 b\log(m_{k-1}))^{-\beta}) + b\log(m_k) H((b\log(m_k))^{-\beta}) + o(\log(m_k))\right)\right) \\
&\quad \geq c_2 k^{-1}\left(1 - \exp\left(-(c_4\log(k-1) - c_5\log k) + o(\log(k))\right)\right),
\end{aligned}$$

for small enough $b > 0$. By choosing N large enough, we can make $c_4 = bc_3 \max_x H(x)$ sufficiently large to ensure that the term $\exp(-(c_4\log(k-1) - c_5\log k) + o(\log(k)) \leqq \frac{1}{2}$ for large k, and hence we have the divergence of the sum in (4.7), giving the result. □

Finally to complete the proof of Theorem 4.4 we apply Lemma 4.5 to a suitable subsequence of the stopping times to show that for N sufficiently large, where $m_k = kN$ we have $T_1^{*m_k} \leqq \lambda_u^{-m_k}(\log(m_k))^{-\beta}$ almost surely. Letting $t_k = T_1^{*m_k}$ we have

$$\log(t_k) \leqq -m_k\log(\lambda_u) - \beta\log(\log(m_k)),$$

and, by taking the inverse, this implies the existence of a constant c_6 such that

$$-m_k \geq \frac{\log(t_k)}{\log(\lambda_u)} + \frac{\beta\log\log\log(\frac{1}{t_k})}{\log(\lambda_u)} + \frac{\log c_6}{\log 2}.$$

Hence

$$|X_{T_1^{*m_k}}| = 2^{-m_k} \geq c_6 t_k^{\log 2/\log \lambda_u} (\log\log(\frac{1}{t_k}))^{1-\log 2/\log \lambda_u}.$$

and hence we have a subsequence which is exceeded infinitely often with probability one. □

For the behavior of the path at arbitrary times, we have

Proposition 4.6. *For any $t, t+h > 0$ and $0 < \delta < 1$, it holds that*

$$P^{*u}[\ |X(t+h) - X(t)| \geq \delta\ ] \leqq C_{2,3} \exp(-C_{2,4}(\frac{\delta |h|^{-\gamma}}{4})^{\frac{1}{1-\gamma}}),$$

where $C_{2,3}$ and $C_{2,4}$ are as in Proposition 4.2.

Proof. It is sufficient to prove the statement for $h > 0$. Choose ω as in the proof of Proposition 4.2. For any given δ, $0 < \delta < 1$, choose $n \in \mathbb{N}$ such that

$$2^{-n+2} \leqq \delta < 2^{-n+3}.$$

Corollary 2.10 (3) implies

$$P^{*u}[\ |X(t+h) - X(t)| \geq \delta,\ t > T_1^{*0}\ ] = 0,$$

and

$$P^{*u}[\ |X(t+h) - X(t)| \geq \delta,\ T_{i-1}^{*n} \leqq t < T_i^{*n} = T_1^{*0},\ \text{for some } i\ ] = 0.$$

In the case that $T_{i-1}^{*n} \leqq t < T_i^{*n} < T_1^{*0}$, if $S_{i+1}^{*n} > h$, then $|X(t+h) - X(t)| < 3 \cdot 2^{-n} < \delta$.

These, combined toghether, imply

$$\begin{aligned} P^{*u}[\ |X(t+h) - X(t)| \geq \delta\ ] &= P^{*u}[\ |X(t+h) - X(t)| \geq \delta,\ T_{i-1}^{*n} \leqq t < T_i^{*n} < T_1^{*0} \text{ for some } i\ ] \\ &\leqq P^{*u}[\ T_{i-1}^{*n} \leqq t < T_i^{*n} < T_1^{*0}\ ,\ S_{i+1}^{*n} < h,\ \text{for some } i\ ] \\ &\leqq P^{*u}[\ S_1^{*n} < h\ ], \end{aligned}$$

and we have the statement from (4.3). □

Proposition 4.6 leads to the Hölder continuity of the paths. Since the proof is standard, we omit it here. (See for example, [7].)

Theorem 4.7. *For any $M > 0$ and any γ' with $0 < \gamma' < \gamma$, the following holds P^{*u}-almost surely. There are positive constants $b = b(\gamma', \omega)$ and $H = H(\gamma', \omega)$ such that*

$$|X(t+h) - X(t)| \leqq b|h|^{\gamma'}, \quad \textit{for any } \ t \in [0, M] \quad \textit{and any } h \in [-H, H].$$

5 Self-repelling processes on $\mathbb{R}$

Here we summarize the basic ingredients of the construction of the corresponding self-repelling processes on $\mathbb{R}$. We start with a sequence of random walks on $\mathbb{Z}$ (instead of the pre-Sierpiński gasket in Section 2). The vertex set that we will use for our walks is $G_n = \{k2^{-n} \mid k = -2^n, -2^n+1, \cdots, 0, 1, 2, \cdots, 2^n\}$.

Remark. We could alternatively consider, for example, $G_n = \{k3^{-n} \mid k = -3^n, -3^n+1, \cdots, 0, 1, 2, \cdots, 3^n\}$. That is, we have the choice of how we divide up the unit interval and any geometric disection into halves, thirds, quarters, etc could be used. The resulting self-repelling processes will be different (even when they are constructed to have the same value of γ). Thus our method produces more than one family of self-repelling processes that continuously interpolate Brownian motion and straight motion on a line. On the Sierpiński gasket, there is an obvious natural unit scale, so our method naturally points at one family of processes. Here we will take the dyadic partition as it is the simplest to work with.

As in Section 2, W_n is the set of continuous functions such that at integer times it takes values in G_n with nearest neighbor jumps from 0 to 1. A sequence of decimation maps Q_k can be defined in a similar way as in the Sierpiński gasket case. The 'reversing number' $N_k(w)$ is now, verbally, the number of points in $G_k \setminus G_{k-1}$ where the decimated walk reverses its jump direction. The 'returning number' retains the same interpretation as before. The generating function $\Phi_n(x,u)$ in (2.2) is much simpler than for the Sierpiński gasket and is given by Proposition 2.1 with

$$\Phi_1(x,u) = \frac{\Psi(x,u)}{1 - 2u\Theta(x,u)}, \quad \Psi(x,u) = x^2, \quad \Theta(x,u) = ux^2.$$

In particular, we have $\Phi_n(x,0) = x^{2^n}$, which implies that when $u = 0$ we have a single path which connects 0 and 2^n by a straight line (i.e., the self-avoiding path on $\mathbb{Z}$), and for $u = 1$ we reproduce the generating function for the simple random walk. In general, $\Phi_n(x,u)$ has non-trivial (x,u) dependence.

We can give explicit formulas for the key quantities $x_u > 0$ and $\lambda_u > 0$ in Proposition 2.3. They are defined by $\Phi(x_u,u) = x_u$ and $\lambda_u = \frac{\partial \Phi}{\partial x}(x_u,u)$ and are given by

$$x_u = \frac{1}{4u^2}(\sqrt{1+8u^2} - 1), \quad \lambda_u = \frac{2}{x_u} = \sqrt{1+8u^2} + 1\,.$$

Note that $x_u > 0$ exists for all $u \geq 0$, and that $\Phi(x,u)$ is regular at x_u. Since for $0 \leqq u < 1$, the paths with a large number of steps are suppressed, we expect that the corresponding walk is self-repelling. For $u > 1$, we expect that the corresponding walk is self-attracting. Also, λ_u, $0 \leqq u \leqq 1$, continuously interpolates $\lambda_0 = 2$ (linear motion) and $\lambda_1 = 4$ (simple random walk). The basic quantities are 'smooth' in the parameter u for all $u \geq 0$. Hence we expect that everything is smooth also for $u > 1$, and as $u \to \infty$ we see that $x_u \to 0$ and $\lambda_u \to \infty$ and the model eventually approaches a completely localized model.

Once we have established these properties of the generating function the subsequent analysis follows quite similar lines to the Sierpiński gasket case in Section 3 and Section 4. For example, the probability measures on the paths are defined by (2.4), and the existence of a continuum limit (Theorem 2.9) and the weak continuity of the path measure P^u in $u \in [0,1]$ (Theorem 3.6) hold. The sample path properties such as Theorem 4.3, Theorem 4.4, and Theorem 4.7 also hold with $\gamma = \frac{\log 2}{\log \lambda_u}$.

References

[1] K. B. Athreya, P. E. Ney, *Branching processes,* Springer, 1972.

[2] M. T. Barlow, E. A. Perkins, *Brownian Motion on the Sierpiński gasket,* Probab. Theor. Relat. Fields **79** (1988) 543–623.

[3] G. Ben Arous, T. Kumagai, *Large deviations of Brownian motion on the Sierpinski gasket* Stoch. Proc. Applic. **85** (2000), 225–235.

[4] J.D. Biggins, N.H. Bingham, *Large deviations in the supercritical branching process*, Adv. Appl. Prob. **25** (1993) 757–772.

[5] E. Bolthausen, *On self-repellent one dimensional random walks,* Probab. Theor. Relat. Fields **86** (1990) 423–441.

[6] D. C. Bryges, G. Slade, *The diffusive phase of a model of self-interacting walks,* Probab. Theor. Relat. Fields **103** (1995) 285–315.

[7] K. Falconer, *Fractal Geometry,* John Wiley and Sons, 1990.

[8] S. Goldstein, *Random walks and diffusion on fractals,* IMA Math. Appl. **8** (1987) 121–129.

[9] A. Greven, F. Hollander, *A variational characterization of the speed of a one-dimensional self-repellent random walk,* Ann. Appl. Probab. **3** (1993) 1067–1099.

[10] B.M. Hambly, O.D. Jones, *Thick and thin points for random recursive fractals*, Preprint, (2000).

[11] B.M. Hambly, T. Kumagai, *Fluctuation of the transition density for Brownian motion on random recursive Sierpinski gaskets*, to appear Stoch. Proc. Applic. (2001).

[12] K. Hattori, *Fractal geometry of self-avoiding processes*, J. Math. Sci. Univ. Tokyo **3** (1996), 379-397.

[13] K. Hattori, T. Hattori, *Self-avoiding process on the Sierpiński gasket,* Probab. Theor. Relat. Fields **88** (1991), 405–528.

[14] K. Hattori, T. Hattori, S. Kusuoka, *Self-avoiding paths on the pre-Sierpiński gasket,* Probab. Theor. Relat. Fields **84** (1990) 1–26.

[15] K. Hattori, T. Hattori, S. Kusuoka, *Self-avoiding paths on the three dimensional Sierpiński gasket,* Publ. RIMS **29** (1993) 455–509.

[16] T. Hattori, S. Kusuoka, *The exponent for mean square displacement of self-avoiding random walk on Sierpiński gasket,* Probab. Theory Relat. Fields **93** (1992) 273–284.

[17] R. van der Hofstad, W. König, *A survey of one-dimensional random polymers*, To appear J. Stat. Phys. (2001).

[18] W. König, *The drift of a one-dimensional self-repellent random walk with bounded increments*, Probab. Theor. Relat. Fields **100** (1994) 513-554.

[19] S. Kusuoka, *A diffusion process on a fractal,* Proc. Taniguchi Symposium (1985) 251–274.

[20] B. Tóth, *'True' self-avoiding walk with generalized bond repulsion on* $\mathbb{Z}$*,* Journ. Stat. Phys. **77** (1994) 17–33.

[21] B. Tóth, *The 'true' self-avoiding walk with bond repulsion on* $\mathbb{Z}$*: Limit theorems,* Ann. Probab. **23** (1995) 1523–1556.

[22] B. Tóth, *Generalized Ray-Knight theory and limit theorems for self-interacting random walks on* $\mathbb{Z}^1$*,* Ann. Probab. **24** (1996) 1324–1367.

[23] B. Tóth, W. Werner, *The true self-repelling motion,* Probab. Theor. Relat. Fields **111** (1998) 375–452.