Weak homogenization of anisotropic diffusion on pre-Sierpiński carpets.

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November 23, 2004

Abstract

We study a kind of 'restoration of isotropy' on the pre-Sierpiński carpet. Let $R_n^x(r)$ and $R_n^y(r)$ be the effective resistances in the x and y directions, respectively, of the Sierpiński carpet at the n-th stage of its construction, if it is made of anisotropic material whose anisotropy is parametrized by the ratio of resistances for a unit square: $r = R_0^y \,/\, R_0^x$. We prove that isotropy is weakly restored asymptotically in the sense that for all sufficiently large n the ratio $R_n^y(r) \,/\, R_n^x(r)$ is bounded by positive constants independent of r. The ratio decays exponentially fast when $r \gg 1$. Furthermore, it is proved that the effective resistances asymptotically grow exponentially with an exponent equal to that found by Barlow and Bass for the isotropic case $r = 1$.

1 Introduction.

In this article we study a kind of homogenization, or restoration of isotropy of anisotropic diffusion, on the pre-Sierpiński carpet [5]. The present work develops ideas arising in two series of recent studies on the diffusion on fractals. One is a study of asymptotically one-dimensional diffusions on Sierpiński gaskets in [9, 10, 8], which contains the discovery of the mechanism on finitely ramified fractals. The other is a detailed study of isotropic diffusion on Sierpiński carpets in [1, 2, 4, 3].

The most interesting aspects of asymptotic behaviors of diffusion (e.g. the spectral dimensions) are embodied in the asymptotic behaviors of effective resistances. A physicist may find it easy to interpret the results on resistances in terms of diffusions. Note (as we will summarize below) that electrical resistance is the rate of heat dissipation caused by electric power. As we will actually use in the proofs, the resistance can be defined as an H_1 norm of electric potential (see (1.1) below), and the potential is a solution to the Laplace equation (a harmonic function) with corresponding Neumann-Dirichlet boundary conditions. Thus it is natural that resistances and diffusions are strongly related. In this paper, rather than going into the relation of the two phenomena in general, we will focus on the behavior of electrical resistances. See [1, 2, 4, 3] on how resistances play an essential part in the construction of diffusions on the Sierpiński carpet, and derivation of their properties.

The Sierpiński carpet is an example of an infinitely ramified fractal [14, 13]. For $n \in \mathbb{Z}_+$ the pre-Sierpiński carpet F_n is the open subset of the unit open square $F_0 = (0,1) \times (0,1)$ obtained by iterating the operation for constructing the Sierpiński carpet, until squares of side length 3^{-n} are reached, where we stop, so that smaller scale structures are absent. The operation is a generalization of that in the construction of the Cantor ternary set: given a square of side length 3^{-m}, we divide it into 9 squares of side length 3^{-m-1}, remove the middle square (with its boundary) and keep the other 8 squares. Thus F_n is an open set in $\mathbb{R}^2$, composed of 8^n squares of side 3^{-n}, and has square shaped holes of side length varying from 3^{-n} to 3^{-1}. It will be convenient later to write $F_n = F_0$ for $n < 0$.

Let $r \in (0,\infty)$, and consider a function $v \in C(\bar{F}_n) \cap H^1(F_n)$, where $C(\bar{F}_n)$ denotes the set of continuous functions on $\bar{F}_n$, and $H^1(F_n)$ the set of square integrable functions whose partial derivatives $\dfrac{\partial v}{\partial x}$ and $\dfrac{\partial v}{\partial y}$ (in the

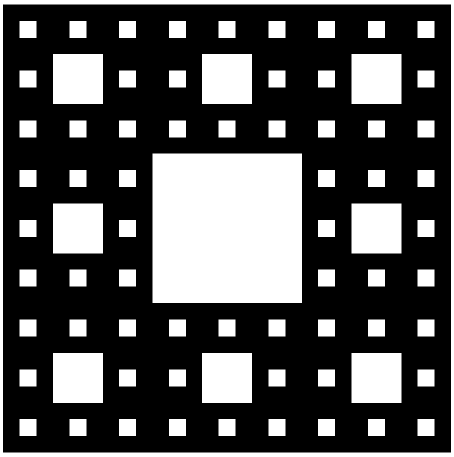

Figure 1: The pre-Sierpiński carpet F_3.

sense of distribution) are square integrable. Put

$$(1.1)\qquad \mathcal{E}_{F_n}(v,v) = \int_{F_n} \left(\left(\frac{\partial v}{\partial x}\right)^2 (x,y) + \frac{1}{r}\left(\frac{\partial v}{\partial y}\right)^2 (x,y) \right) dx\,dy\,.$$

In physical terms $\mathcal{E}_{F_n}(v,v)$ is the rate of energy dissipation for the potential (voltage) distribution v if F_n is made of a material with a uniform but anisotropic electrical resistivity, with anisotropy parameter r. For a unit square made of this material, the total resistance is 1 in the x-direction and r in the y-direction, and the principal axes of the resistivity tensor are parallel to the x and y axes.

Define $R_n^x(r)$, the effective resistance of F_n in the x direction, by the following (principle of minimum heat production):

$$(1.2)\qquad \frac{1}{R_n^x(r)} = \inf\left\{\mathcal{E}_{F_n}(v,v)\right\}\,,$$

where the infimum is taken over all the functions $v \in C(\bar{F}_n) \cap H^1(F_n)$, satisfying boundary conditions

$$(1.3)\qquad v(0,y) = 0,\ v(1,y) = 1,\quad 0 \le y \le 1.$$

The effective resistance in the y direction $R_n^y(r)$ is defined in a similar manner, with boundary conditions

$$v(x,0)=0,\ v(x,1)=1,\quad 0\le x\le 1. \tag{1.4}$$

Obviously,

$$R_0^x(r)=1 \quad \text{and} \quad R_0^y(r)=r\,. \tag{1.5}$$

Set

$$H_n(r)=\frac{R_n^y(r)}{R_n^x(r)}\,; \tag{1.6}$$

thus $H_n(r)$ measures the effective anisotropy of F_n if it is composed of material with anisotropy parameter r. It is easy (see Lemma 3.1) to verify that

$$R_n^x(r)=rR_n^y(1/r), \qquad H_n(r)=H_n(1/r)^{-1}.$$

We have the following conjecture:

Conjecture. *('Strong Homogenization').*

$$\lim_{n\to\infty} H_n(r)=1,\ \text{for each } r\in(0,\infty). \tag{1.7}$$

In this paper, we prove the following weak homogenization property:

Theorem 1.1. *There exists a constant $1\le K<\infty$ such that*

$$K^{-1}\le \liminf_{n\to\infty} H_n(r)\le \limsup_{n\to\infty} H_n(r)\le K \quad \textit{for each } r\in(0,\infty)\,.$$

Our proof gives explicit bounds: we can take $K=6333$, which may be compared with the conjectured value $K=1$ in (1.7). (Our bounds, and proof, have improved since we announced them in [5].)

Theorem 1.1 does not give information on the asymptotic behavior in n of $R_n^x(r)$ and $R_n^y(r)$. However, we have the following result:

Theorem 1.2. *For each $r>0$,*

$$0<\inf_n \rho^{-n}R_n^z(r)\le \sup_n \rho^{-n}R_n^z(r)<\infty\,,\quad z=x,y,$$

where ρ is the growth exponent for the isotropic case $r=1$ given in [2, 4].

Thus the effective resistances $R_n^x(r)$ and $R_n^y(r)$ both grow asymptotically like ρ^n, and so the growth exponent ρ found in [2] is universal in the sense that it is independent of the anisotropy r.

We see from (1.5) and (1.6) that $H_0(r) = r$. Thus Theorem 1.1 implies that if $r \gg 1$, $H_n(r)$ should be relatively small when n is large. In fact, we have the following estimate for the decrease of H_n in n.

Theorem 1.3. *There exist constants* $c \in (0, \infty)$, $s_1 \in (0, 1)$ *such that*

$$1 \le s^{-1} H_n((9/7)^n s) \le \exp(cs^{-\xi}), \quad n \ge 1, s \ge s_1, \tag{1.8}$$

where $\xi = \log 2/\log 7$. *In particular*

$$\lim_{s\to\infty} \liminf_{n\to\infty} s^{-1} H_n((9/7)^n s) = \lim_{s\to\infty} \limsup_{n\to\infty} s^{-1} H_n((9/7)^n s) = 1\,.$$

Thus when $s = (7/9)^n r$ is large, $H_n(r) \approx (7/9)^n r$. A similar result holds for small s:

$$\lim_{s\to 0} \limsup_{n\to\infty} s^{-1} H_n((7/9)^n s) = \lim_{s\to 0} \liminf_{n\to\infty} s^{-1} H_n((7/9)^n s) = 1\,.$$

We can also obtain scaling relations of this kind for the effective resistances $R_n^x(r)$ and $R_n^y(r)$ – see the proof of the theorem. Our proof also implies that $\lim_{r\to\infty} R_n^x(r) = (3/2)^n$ and $\lim_{r\to\infty} r^{-1} R_n^y(r) = (7/6)^n$. (See (3.22) and (3.20).) Therefore

$$\lim_{r\to\infty} r^{-1} H_n(r) = \left(\frac{7}{9}\right)^n, \quad n \ge 0. \tag{1.9}$$

We have no proof of the existence of the scaling limit

$$h(s) = \lim_{n\to\infty} s^{-1} H_n((9/7)^n s)\,,$$

but Theorem 1.3 implies that if h does exist then $\lim_{s\to\infty} h(s) = 1$. For further comments and conjectures on the form of h see [5].

Proofs of Theorem 1.1 and Theorem 1.3 are given in Section 3. The basic tools to prove Theorem 1.1 are Propositions 3.2 and 3.3, which are recursive inequalities for the effective resistances, which give good bounds in the anisotropic regime, that is when $H_n(r)$ is very different from 1. If $H_n(r) \gg 1$, then, roughly speaking, these inequalities state that the smaller effective resistance $R_n^x(r)$ grows as $(3/2)^n$, while the larger effective resistance $R_n^y(r)$ grows as $(7/6)^n r$. So as long as $H_n(r) \gg 1$, we have $H_n(r) \approx (7/9)^n r$, and thus $H_n(r)$ approaches 1 exponentially fast. Theorem 1.3 shows that we can make precise this argument on the exponential decay of $H_n(r)$. In fact, the estimates in Propositions 3.2 and 3.3 are precise enough to allow us to prove that $H_n(r)$ is bounded for all large n, so proving Theorem 1.1.

We prove Theorem 1.2 in Section 4, by giving another recursive inequality (Proposition 4.1), analogous to those given in [2] for $r = 1$. Section 2 is devoted to basic estimates used both in Section 3 and Section 4.

A strong homogenization result similar to (1.7) is proved in [9, 10, 5] for the pre-Sierpiński gasket, using explicit renormalization group recursion relations for quantities analogous to $R_n^x(r)$ and $R_n^y(r)$. As the Sierpiński gasket is finitely ramified, these recursion relations are finite dimensional, and so exact calculations are possible. We expect that this kind of restoration of isotropy will occur on a wide class of fractals – see [5]. That this is difficult to prove for the Sierpiński carpet reflects the fact that it is an infinitely ramified fractal, and so the renormalization group recursion acts on an infinite dimensional space. The rigorous inequalities in Propositions 3.2, 3.3, and 4.1 provide a version of the renormalization group relations.

We conclude this section with some remarks.

1. With the change of the coordinate $y' = \sqrt{r}\, y$, the defining equation (1.2) has an isotropic expression, so our results also apply to rectangular boards made of isotropic material.

2. F_n is contained in the unit square and the unit structure is of order 3^{-n}. But the scale invariance of resistance in two dimensions implies that the effective resistances are the same if we defined F_n as a figure with unit structures of order 1 and of total size $3^n \times 3^n$; i.e., constructing the figure outward instead of inward. The results in this paper hold as they are, with only minor notational changes.

3. Analogous results can also be obtained for the cross-wire networks G_n introduced in [2]. The network G_n is obtained from F_n by replacing each of the 8^n squares of side 3^{-n} in F_n by a horizontal and vertical crosswire of four linear resistors (joined at the center of the square), where each horizontal resistor has resistance $1/2$ and each vertical resistor has resistance $r/2$. (See [7] for basic facts about resistor networks.) The results in this paper hold as they are, with similar proofs.

4. Our proofs should also be effective for the class of 'generalized Sierpiński carpets' considered in [2, Eq. (3.1)]. In particular, with only minor

changes, they apply to (k,l) – Sierpiński carpets. Here the sets F_n are constructed recursively by dividing each square of side $k^{-(n-1)}$ in F_{n-1} into k^2 squares, and throwing out a block of ℓ^2 squares at the center. (We take $k \geq 3$ and $k > \ell$.) The numbers appearing in the results, such as the exponents $7/9$, $7/6$, $3/2$, and ρ, will of course in general be different for different figures.

5. The proof of the conjecture (1.7) seems to us to be quite hard. We suspect that it is similar in difficulty to the problem of improving the inequalities

$$\frac{1}{4}\rho^n \leq R_n^x(1) \leq 4\rho^n, \qquad n \geq 0,$$

given in [2], to proving the existence of the conjectured limit

$$\lim_{n\to\infty} \rho^{-n} R_n^x(1)\,.$$

Acknowledgements. The research of M. T. Barlow is supported by a NSERC (Canada) grant. The research of T. Hattori is supported in part by a Grant-in-Aid for Scientific Research (C) from the Ministry of Education, Science, Sports and Culture.

2 Basic estimates on energy of harmonic functions.

Throughout this section, we fix $r > 0$ and $n \in \mathbb{Z}$.

The first two Propositions deal with the principle of minimum heat production in terms of potentials and currents, respectively. They are straightforward extensions of the isotropic case $r = 1$ in [2], to which we refer for a proof.

Proposition 2.1. *There exists a unique function $v = V_n^x(r)$ (or $V_n^y(r)$) in $C(\bar{F}_n) \cap H^1(F_n)$ with $\nabla v \in L^2(\partial F)$ which attains the infimum of (1.2) with the boundary condition (1.3) (or (1.4), respectively);*

$$R_n^x(r)^{-1} = \mathcal{E}_{F_n}(V_n^x(r), V_n^x(r)), \quad R_n^y(r)^{-1} = \mathcal{E}_{F_n}(V_n^y(r), V_n^y(r)). \tag{2.1}$$

The functions satisfy the following Laplace equation on F_n

$$\frac{\partial^2 v}{\partial x^2}(x,y) + \frac{1}{r}\frac{\partial^2 v}{\partial y^2}(x,y) = 0\,, \;\; (x,y) \in F_n\,, \tag{2.2}$$

with boundary conditions (1.3) (or (1.4), respectively), and Neumann boundary conditions $\dfrac{\partial v}{\partial n} = 0$*, on the rest of* ∂F_n*, except at the corners of the squares in* ∂F_n*. In particular, for* $z = x, y$*,*

$$0 \le V_n^z(r)(x,y) \le 1, \quad (x,y) \in \bar{F}_n\,. \tag{2.3}$$

Note also that the symmetry of F_n implies

$$\begin{aligned} V_n^x(r)(x,y) &= V_n^x(r)(x,1-y), && 0 \le x \le 1,\ 0 \le y \le 1, \\ V_n^x(r)(x,y) + V_n^x(r)(1-x,y) &= 1, && 0 \le x \le 1,\ 0 \le y \le 1, \end{aligned} \tag{2.4}$$

with similar relations for $V_n^y(r)$.

There is a dual formulation of resistance in terms of currents. Denote by $\mathcal{C}(F_n)$, the set of $\mathbb{R}^2$ valued square integrable functions $j \in BV(F_n)$ (integrable functions whose derivatives in the sense of distribution are measures with finite total variations [15]), satisfying current conservation $\mathrm{div} j = 0$ (in the sense of distribution). We call an element $j = (j_x, j_y)$ of $\mathcal{C}(F_n)$, a current on F_n.

Remark. Note that as j is defined on the open set F_n, the values of j on ∂F_n are not defined. However, we will need to express the resistance $R_n^x(r)$ in terms of the minimum energy of a current j with total flux 1 across F_n, and to define the class of feasible currents for this optimization problem we need to consider boundary values for currents $j \in \mathcal{C}(F_n)$. If $j \in BV(F_n)$ then by [12, p.325] the *rough trace* j^* exists on ∂F_n. For the precise definition of j^* see [12] – but note from [12] that if $(x_0, y_0) \in \partial F_n$ then

$$j^*(x_0, y_0) = \lim_{(x,y) \to (x_0, y_0)} j(x,y)$$

whenever this limit exists. Thus, essentially, for a well behaved function the rough trace is simply a continuous extension to the boundary. A general version of the Gauss–Green formula [12, p.340] expresses an integration of j over the domain F_n by a contour integration of j^* along ∂F_n. The currents we will consider in this paper have analytic continuations to ∂F_n, except at a finite number of points. (See the proof of Lemma 2.8 in Appendix A.) Thus we can consistently extend j to the boundary ∂F_n, and from now on we will do so whenever necessary without further comment.

For a vector field $j = (j_x, j_y) \in L^2(F_n)$, and $B \subset F_n$, define

$$E_B(j,j) = \int_B (j_x^2(x,y) + r\, j_y^2(x,y))\, dx\, dy\,.$$

Proposition 2.2.

$$R_n^x(r) = \inf \{E_{F_n}(j,j)\} \; , \tag{2.5}$$

where the infimum is taken over all $j = (j_x, j_y) \in \mathcal{C}(F_n)$ which satisfy $j \cdot n = 0$, a.e., on the boundary of F_n, except at two edges $x = 0$ and $x = 1$, where we impose

$$\int_0^1 j_x(0,y)\, dy = \int_0^1 j_x(1,y)\, dy = -1 \, . \tag{2.6}$$

Here n is the unit normal vector at the boundary of F_n, and $j \cdot n$ denotes inner product of vectors. The function $j = J_n^x(r)$ which attains the infimum of (2.5) exists and is unique, and is given by

$$J_n^x(r) = (J_{nx}^x(r), J_{ny}^x(r)) = -\left(R_n^x(r) \frac{\partial V_n^x(r)}{\partial x} \, , \; \frac{1}{r} R_n^x(r) \frac{\partial V_n^x(r)}{\partial y} \right) . \tag{2.7}$$

Similarly, there exists a unique function $J_n^y(r) \in \mathcal{C}(F_n)$ which satisfies

$$R_n^y(r) = \inf \{E_{F_n}(j,j)\} = E_{F_n}(J_n^y(r), J_n^y(r)) \, , \tag{2.8}$$

where j satisfies similar conditions as before, with

$$\int_0^1 j_y(x,0)\, dx = \int_0^1 j_y(x,1)\, dx = -1. \tag{2.9}$$

in place of (2.6).

Remark. The minus sign in (2.7) comes from the sign conventions in the boundary conditions (1.3) and (2.6), which are the traditions in the study of electricity. It is a well-known historical misfortune that not only do we need minus signs here, but the electrons in reality move in opposite direction to the currents when they are defined in this way.

Remark. We can regard Proposition 2.1 and Proposition 2.2 as giving $R_n^x(r)$ in terms of an optimization problem and its dual. In view of this, we will use the language of optimization theory and refer, for example, to a flow which satisfies the conditions of Proposition 2.2 as a feasible flow.

Remark. If $n < 0$, then since $F_n = F_0$, we have $V_n^x = V_0^x$, $J_n^x = J_0^x$, etc.

Note that (2.3) and (2.7) imply that

$$J^x_{nx}(r)(0,y) \geq 0, \quad 0 \leq y \leq 1,$$

while the symmetry of F_n implies

$$(2.10)\; J^x_{nx}(0,y) = J^x_{nx}(1,y) = J^x_{nx}(0,1-y) = J^x_{nx}(1,1-y),\ 0 \leq y \leq 1,$$

with similar relations for $J^y_n(r)$.

Next we turn to a couple of basic estimates of the energy in terms of potentials and currents.

Definition 2.3. *For $G \subset F_n$, define the bilinear form*

$$\mathcal{E}_G(f,g) = \int_G \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{1}{r}\frac{\partial f}{\partial y}\frac{\partial g}{\partial y} \right) dx\, dy\,, \quad f,g \in C(\bar{G}) \cap H^1(G).$$

Thus $\mathcal{E}_G$ is the Dirichlet form associated with the self-adjoint operator

$$\mathcal{L} = \frac{\partial^2}{\partial x^2} + \frac{1}{r}\frac{\partial^2}{\partial y^2}$$

on the space $L^2(G,\mu)$. (Here μ is Lebesgue measure).

The following Lemma is an application of Cauchy-Schwarz. We write 1_G for the indicator function of G, and $||\cdot||_\infty$ for the L^∞ norm.

Lemma 2.4. *Let $f,\ g \in C(\bar{F}_n) \cap H^1(F_n)$. Then*

$$\mathcal{E}_G(f\,g, f\,g) \leq 2||g 1_G||^2_\infty\ \mathcal{E}_G(f,f) + 2||f 1_G||^2_\infty\ \mathcal{E}_G(g,g).$$

Proof. Write (just for now) $f_x = \dfrac{\partial f}{\partial x}$. Note that $((fg)_x)^2 = (fg_x + f_x g)^2 \leq 2f^2 g_x^2 + 2g^2 f_x^2$. So,

$$\begin{aligned} \mathcal{E}_G(fg,fg) &\leq 2\int_G f^2(g_x^2 + r^{-1}g_y^2)\,dx\,dy + 2\int_G g^2(f_x^2 + r^{-1}f_y^2)\,dx\,dy \\ &\leq 2||f1_G||^2_\infty\ \mathcal{E}_G(g,g) + 2||g1_G||^2_\infty\ \mathcal{E}_G(f,f). \end{aligned}$$

□

Definition 2.5. *Let $n \geq 0$, $m \geq 0$. Set*

$$\begin{aligned} B^x_{m,i} &= [0,1] \times [i\,3^{-m}, (i+1)\,3^{-m}], \quad 0 \leq i \leq 3^m - 1, \\ B^y_{m,i} &= [i\,3^{-m}, (i+1)\,3^{-m}] \times [0,1], \quad 0 \leq i \leq 3^m - 1. \end{aligned}$$

We now estimate the energy associated with the potential $V_n^x(r)$ in the thin strip $B_{m,0}^x$, which lies adjacent to the x-axis. To avoid too many subscripts we will sometimes write $\mathcal{E}[G](f,f,)$ for $\mathcal{E}_G(f,f,)$ in what follows.

Lemma 2.6. *For $m, n \geq 0$,*

$$\mathcal{E}[B_{m,0}^x \cap F_n](V_n^x(r), V_n^x(r)) \leq 2^{-m} R_n^x(r)^{-1}, \tag{2.11}$$

$$\mathcal{E}[B_{m,0}^y \cap F_n](V_n^y(r), V_n^y(r)) \leq 2^{-m} R_n^y(r)^{-1}. \tag{2.12}$$

Proof. Write

$$E_{m,i} = \mathcal{E}[B_{m,i}^x \cap F_n](V_n^x, V_n^x),$$

set

$$\tilde{B}_{m,i}^x = [0,1] \times [i2^{-1}3^{-m+1}, (i+1)2^{-1}3^{-m+1}] \quad i = 0, 1,$$

and let

$$\tilde{E}_{m,i} = \mathcal{E}[\tilde{B}_{m,i}^x \cap F_n](V_n^x, V_n^x), \quad i = 0, 1.$$

Thus we have

$$E_{m,0} = \sum_{j=0}^{2} E_{m+1,j} = \sum_{j=0}^{1} \tilde{E}_{m+1,j}.$$

For $m \in \mathbb{Z}_+$, define a potential $v \in C(\bar{F}_n) \cap H^1(F_n)$ by

$$v(x,y) = \begin{cases} V_n^x(r)(x, 3^{-m} - y), & (x,y) \in \tilde{B}_{m+1,0}^x, \\ V_n^x(r)(x,y), & (x,y) \in F_n \setminus \tilde{B}_{m+1,0}^x. \end{cases}$$

As v satisfies the boundary condition (1.3), (2.1), (1.2), and the definition of v imply that

$$\sum_{i=0}^{3^{m+1}-1} E_{m+1,i} = R_n^x(r)^{-1} \leq \mathcal{E}_{F_n}(v,v) = 2\tilde{E}_{m+1,1} + \sum_{i=3}^{3^{m+1}-1} E_{m+1,i}. \tag{2.13}$$

Therefore $\tilde{E}_{m+1,0} \leq \tilde{E}_{m+1,1}$, and so $2\tilde{E}_{m+1,0} \leq \tilde{E}_{m+1,0} + \tilde{E}_{m+1,1} = E_{m,0}$. As $B_{m+1,0}^x \subset \tilde{B}_{m+1,0}^x$ this implies that

$$E_{m+1,0} \leq \tilde{E}_{m+1,0} \leq \tfrac{1}{2} E_{m,0}.$$

Iterating, and using the fact that $E_{0,0} = R_n^x(r)^{-1}$, we obtain (2.11). (2.12) follows by interchanging x and y axes. □

We have a corresponding result for currents.

Lemma 2.7. *For $n, m \geq 0$,*

$$E[B^y_{m,0} \cap F_n](J^x_n(r), J^x_n(r)) \leq 2^{-m} R^x_n(r), \tag{2.14}$$

$$E[B^x_{m,0} \cap F_n](J^y_n(r), J^y_n(r)) \leq 2^{-m} R^y_n(r). \tag{2.15}$$

Proof. Let

$$\begin{aligned}
E'_{m,i} &= E[B^y_{m,i} \cap F_n](J^x_n(r), J^x_n(r))\,, \\
\tilde{B}^y_{m,i} &= [i2^{-1}3^{-m+1}, (i+1)2^{-1}3^{-m+1}] \times [0,1], \quad i = 0,1, \\
\tilde{E}'_{m,i} &= E[\tilde{B}^y_{m,i} \cap F_n](J^x_n(r), J^x_n(r)).
\end{aligned}$$

So, as before, we have

$$E'_{m,0} = \sum_{j=0}^{2} E'_{m+1,j} = \sum_{j=0}^{1} \tilde{E}'_{m+1,j}.$$

For $m \in \mathbb{Z}_+$, define a current $j \in \mathcal{C}(F_n)$ by

$$j(x,y) = \begin{cases} (J^x_{nx}(r), -J^x_{ny}(r))(3^{-m} - x, y)\,, & (x,y) \in \tilde{B}^y_{m+1,0}\,, \\ J^x_n(r)(x,y)\,, & (x,y) \in F_n \setminus \tilde{B}^y_{m+1,0}\,. \end{cases}$$

It is straightforward to check that $j \in \mathcal{C}(F_n)$ and satisfies (2.6). Therefore

$$\sum_{i=0}^{3^{m+1}-1} E'_{m+1,i} = R^x_n(r) \leq E(j,j) = 2\tilde{E}'_{m+1,1} + \sum_{i=3}^{3^{m+1}-1} E'_{m+1,i}\,, \tag{2.16}$$

and the remainder of the proof proceeds as in Lemma 2.6. □

The next lemma will play a crucial role when we obtain an upper bound on quantities like $R^x_n(r)$ by constructing a 'feasible flow' $j \in \mathcal{C}(F_n)$ and using the energy-minimizing principle (2.5). Except in the simplest cases, this construction requires estimates on the energy of a current which can 'turn corners'.

Fix (for now) $n, m \geq 0$, $r > 0$, and let $G = B^y_{m,0} \cap F_n$. Let R_G be the resistance of G between the lines $y = 0$ and $y = 1$. We define (and calculate), R_G by the methods of Propositions 2.1 and 2.2. Thus

$$R_G = \inf\{E_G(j,j)\} \tag{2.17}$$

where the infimum is over currents j on G satisfying the boundary conditions

$$\int_0^{3^{-m}} j_y(x,0)dx = \int_0^{3^{-m}} j_y(x,1)dx = -1,$$

and $j \cdot n = 0$ a.e. on the remainder of the boundary of G. As G consists of 3^m scaled copies of F_{n-m}, it is easy to see that the infimum in (2.17) is attained by the current $\tilde{J}$ obtained by piecing together 3^m scaled copies of $J^y_{n-m}(r)$:

$$\tilde{J}(x,y) = 3^m J^y_{n-m}(3^m x, 3^m y - [3^m y])(r), \quad (x,y) \in G,$$

Here $[3^m y]$ is the largest integer less than or equal to $3^m y$. Therefore

$$R_G = E_G(\tilde{J},\tilde{J}) = 3^m R^y_{n-m}(r).$$

The following result is proved in the Appendix.

Lemma 2.8. *There exists $L = L^{(n,m)} \in BV(F_n) \cap L^2(F_n)$ satisfying*

$$
\begin{aligned}
&(2.18) & \mathrm{div}(L) &= 0 && \textit{(as a distribution) on } G,\\
&(2.19) & L &= 0 && \textit{on } F_n - \bar{G},\\
&(2.20) & L &= J^x_n(r) && \textit{in a neighborhood of } \{x=0, 0<y<1\},\\
&(2.21) & L &= -\tilde{J} && \textit{in a neighborhood of } \{0<x<3^{-m}, y=0\},\\
&(2.22) & \frac{\partial L}{\partial n} &= 0 && \text{a.e. } \textit{on the remainder of the boundary of } G,
\end{aligned}
$$

such that

$$
\begin{aligned}
E_G(L,L) &\le E_G(J^x_n(r), J^x_n(r)) + E_G(\tilde{J},\tilde{J})\\
&\le 2^{-m} R^x_n(r) + 3^m R^y_{n-m}(r).
\end{aligned}
$$

The current L constructed in Lemma 2.8 provides a current which has total flux 1 across G coming in from left edge $x = 0$ and going out at bottom edge $y = 0$. L will be considered as a part of a current in the larger domain in such a way that the boundary condition (current conservation at the boundary of G) specified by (2.20) and (2.21) must be satisfied.

3 Recursion relations effective in the anisotropic regime.

3.1 Basic tools.

We begin with some elementary observations.

Lemma 3.1. *For $r \in (0,\infty)$ and $n \in \mathbb{Z}_+$,*

$$R_n^x(r) = rR_n^y(1/r),$$
$$H_n(r) = H_n(1/r)^{-1}.$$

Proof. Fix n, and write $S^x(a,b)$ for the resistance in the x direction of F_n, if it is composed of anisotropic material with resistivity a in the x direction, and b in the y direction, and define $S^y(a,b)$ analogously. Then $S^x(a,b) = S^y(b,a)$, $R_n^x(r) = S^x(1,r)$, $R_n^y(r) = S^y(1,r)$, $S^x(\lambda a, \lambda b) = \lambda S^x(a,b)$, and so

$$R_n^x(r) = S^x(1,r) = S^y(r,1) = rS^y(1,r^{-1}) = rR_n^y(r^{-1}).$$

Also,

$$H_n(r) = \frac{S^y(1,r)}{S^x(1,r)} = \frac{S^x(r,1)}{S^y(r,1)} = \frac{1}{H_n(r^{-1})}.$$

□

The following two propositions give the recursion relations which are the essential tools for this section.

Proposition 3.2. *Let $r > 0$, $n \geq 1$, and $m \geq 2$. Then*

$$R_{n-1}^x(r)^{-1} \leq \frac{3}{2}R_n^x(r)^{-1} \leq (1+\frac{a_1}{2^m})R_{n-1}^x(r)^{-1} + A_1 3^m R_{n-m}^y(r)^{-1}, \tag{3.1}$$

$$R_{n-1}^y(r)^{-1} \leq \frac{3}{2}\,R_n^y(r)^{-1} \leq (1+\frac{a_1}{2^m})\,R_{n-1}^y(r)^{-1} + A_1 3^m R_{n-m}^x(r)^{-1}, \tag{3.2}$$

where $a_1 = 8/3$, $A_1 = 4/9$.

Proposition 3.3. *Let $r > 0$, $n \geq 1$, and $m \geq 2$. Then*

$$R_{n-1}^x(r) \leq \frac{6}{7}\,R_n^x(r) \leq (1+\frac{a_2}{2^m})\,R_{n-1}^x(r) + A_2 3^m R_{n-m}^y(r)\,, \tag{3.3}$$

$$R_{n-1}^y(r) \leq \frac{6}{7}\,R_n^y(r) \leq (1+\frac{a_2}{2^m})\,R_{n-1}^y(r) + A_2 3^m R_{n-m}^x(r)\,, \tag{3.4}$$

where $a_2 = 16/7$, $A_2 = 4/21$.

Remark. (3.1) and (3.3) are good bounds when $H_n(r) \gg 1$, while (3.2) and (3.4) are good when $H_n(r) \ll 1$.

While we have, for clarity, given four separate inequalities, (3.2) and (3.4) are immediate consequences of (3.1), (3.3) and Lemma 3.1. So we need only prove (3.1) and (3.3).

Definition 3.4. *Denote the eight scaled copies of $\bar{F}_{n-1}$ which compose $\bar{F}_n$, by*

$$A_{ij} = ([i/3,\ (i+1)/3] \times [j/3,\ (j+1)/3]) \cap \bar{F}_n\,, \quad (i,j) \in \{0,1,2\}^2 \setminus \{(1,1)\}\,.$$

The left hand side inequalities in Propositions 3.2 and 3.3 are easy – this is essentially just a standard argument involving shorts and cuts. See [7], [6].

Proof of the left hand side of (3.1) . Define a potential $v \in C(\bar{F}_n) \cap H^1(F_n)$ by

$$v(x,y) = \begin{cases} \frac{2}{7} V^x_{n-1}(r)(3x, 3y-j)\,, & (x,y) \in A_{0j},\ j = 0,1,2, \\ \frac{2}{7} + \frac{3}{7} V^x_{n-1}(r)(3x-1, 3y-j)\,, & (x,y) \in A_{1j},\ j = 0,2, \\ \frac{5}{7} + \frac{2}{7} V^x_{n-1}(r)(3x-2, 3y-j)\,, & (x,y) \in A_{2j},\ j = 0,1,2. \end{cases}$$

Then v is continuous, and using (1.2) we have

$$\begin{aligned} R^x_n(r)^{-1} &\le \mathcal{E}(v,v) \\ &\le 6\mathcal{E}_{F_{n-1}}(\frac{2}{7}V^x_{n-1}(r), \frac{2}{7}V^x_{n-1}(r)) + 2\mathcal{E}_{F_{n-1}}(\frac{3}{7}V^x_{n-1}(r), \frac{3}{7}V^x_{n-1}(r)) \\ &= \frac{6}{7} R^x_{n-1}(r)^{-1}. \end{aligned}$$

□

Proof of the left-hand side of (3.3) Define a current $j \in \mathcal{C}(F_n)$ by

$$j(x,y) = \begin{cases} \frac{3}{2} J^x_{n-1}(r)(3x-i, 3y-j)\,, & (x,y) \in A_{ij}\,,\ i = 0,1,2,\ j = 0,2, \\ 0\,, & (x,y) \in A_{01} \cup A_{21}\,. \end{cases}$$

Then it is easy to check that j satisfies the current conservation and the boundary conditions given in Proposition 2.2, so that by (2.5), and the fact that $A_{ij} \cong 3^{-1} F_{n-1}$, we have

$$R^x_n(r) \le E(j,j) = 6E_{F_{n-1}}(\frac{1}{2}J^x_{n-1}(r), \frac{1}{2}J^x_{n-1}(r)) = \frac{3}{2}\, R^x_{n-1}(r)\,.$$

□

The proofs of the right hand side inequalities in Propositions 3.2 and 3.3 are more involved.

Proof of the right hand side of (3.1). As r will be fixed throughout this proof, we will simplify notation by writing $V_n^x = V_n^x(r)$, $J_n^x = J_n^x(r)$, etc. Fix $n \geq 0$, $m \geq 2$, set $k = n - m$ and recall our convention that $V_k^x = V_0^x$ if $k < 0$.

Set

$$\varphi(x,y) = \frac{1}{3} V_{n-1}^x(3x - i, 3y - j) + \frac{i}{3}, \qquad \text{if } (x,y) \in A_{ij},\ (i,j) \neq (1,1).$$

Note that $\varphi \in C(\bar{F}_n) \cap H^1(F_n)$ and

$$\mathcal{E}[A_{ij} \cap F_n](\varphi,\varphi) = \frac{1}{9}(R_n^x)^{-1} \qquad \text{for } (i,j) \neq (1,1).$$

Now let

$$\psi(x,y) = \begin{cases} 1, & (x,y) \in \bar{F}_n \setminus (A_{01} \cup A_{21}), \\ 0, & (x,y) \in (A_{01} \cup A_{21}) \setminus (B^x_{m,3^{m-1}} \cup B^x_{m,23^{m-1}-1}), \\ V^y_{n-m}(3^m x - [3^m x], 3^m(y - \frac{1}{3})), & (x,y) \in (A_{01} \cup A_{21}) \cap B^x_{m,3^{m-1}}, \\ V^y_{n-m}(3^m x - [3^m x], 3^m(\frac{2}{3} - y)), & (x,y) \in (A_{01} \cup A_{21}) \cap B^x_{m,23^{m-1}-1}. \end{cases}$$

We can check that ψ is continuous, and so $\psi \in C(\bar{F}_n) \cap H^1(F_n)$. Note that φ, ψ are symmetric about the line $y = \frac{1}{2}$, and that

$$\varphi(x,y) + \varphi(1-x,y) = 1, \quad \psi(x,y) = \psi(1-x,y). \tag{3.5}$$

Set

$$v(x,y) = \begin{cases} \varphi(x,y)\,\psi(x,y), & 0 < x \leq \frac{1}{2},\ 0 \leq y \leq 1, \\ 1 - (1 - \varphi(x,y))\,\psi(x,y), & \frac{1}{2} \leq x \leq 1,\ 0 \leq y \leq 1. \end{cases}$$

Continuity of v follows from that of φ and ψ, and (3.5); thus $v \in C(\bar{F}_n) \cap H^1(F_n)$. It is also easy to see that v satisfies the boundary conditions (1.3). Noting that $A_{ij} \cong 3^{-1}F_{n-1}$ and using the symmetry of v, we have

$$(R_n^x)^{-1} \leq \mathcal{E}_{F_n}(v,v) = 4\mathcal{E}[A_{00} \cap F_n](v,v) + 2\mathcal{E}[A_{01} \cap F_n](v,v) + 2\mathcal{E}[A_{10} \cap F_n](v,v). \tag{3.6}$$

As $\psi = 1$ on $A_{00} \cup A_{10}$ we have for $j = 0, 1$,

$$\mathcal{E}[A_{0j} \cap F_n](v,v) = \mathcal{E}[A_{00} \cap F_n](\varphi,\varphi) = \frac{1}{9} R_{n-1}^x(r)^{-1}. \tag{3.7}$$

Now set $G = [0, \frac{1}{3}] \times [\frac{1}{3}, \frac{1}{3} + 3^{-m}]$. As $\psi = 0$ on $(A_{10} - G) \cap \{y < \frac{1}{2}\}$, by the symmetry of G and Lemma 2.4,

$$\begin{aligned} \mathcal{E}[A_{10}](v,v) &= 2\mathcal{E}_G(\varphi\psi, \varphi\psi) \\ &\leq 4\|\psi 1_G\|_\infty^2 \mathcal{E}_G(\varphi,\varphi) + 4\|\varphi 1_G\|_\infty^2 \mathcal{E}_G(\psi,\psi) \\ &= 4\mathcal{E}_G(\varphi,\varphi) + \frac{4}{9}\mathcal{E}_G(\psi,\psi). \end{aligned} \tag{3.8}$$

Using scaling and Lemma 2.6,

$$\mathcal{E}_G(\varphi,\varphi) = \frac{1}{9}\mathcal{E}[B^x_{m-1,0} \cap F_{n-1}](V^x_{n-1}, V^x_{n-1}) \leq \frac{2^{1-m}}{9}(R^x_{n-1})^{-1}, \tag{3.9}$$

while as G consists of 3^{m-1} segments, each congruent to $3^{-m}F_{n-m}$,

$$\mathcal{E}_G(\psi,\psi) = 3^{m-1}\mathcal{E}(V^y_{n-m}, V^y_{n-m}) = 3^{m-1}(R^y_{n-m})^{-1}. \tag{3.10}$$

Combining (3.6), (3.7), (3.8), (3.9), and (3.10) we deduce that

$$\begin{aligned} (R^x_n)^{-1} &\leq \frac{2}{3}\Big((R^x_{n-1})^{-1} + 2\Big(\frac{4}{9}2^{-m+1}(R^x_{n-1})^{-1} + \frac{4}{9}3^{m-1}(R^y_{n-m})^{-1}\Big)\Big) \\ &= \frac{2}{3}\Big((R^x_{n-1})^{-1}(1 + \frac{8}{3}2^{-m}) + \frac{4}{9}3^m(R^y_{n-m})^{-1}\Big). \end{aligned}$$

□

Proof of the right hand side of (3.3). This proof uses similar ideas to the one given above, but as we have to work with currents rather than potentials, it is a bit more complicated. Define a vector field K^1 on F_n by

$$K^1(x,y) = \begin{cases} J^x_n(3x-i, 3y-j), & (x,y) \in A_{ij}, i = 0,2,\ 0 \leq j \leq 2, \\ \frac{3}{2}J^x_n(3x-i, 3y-j), & (x,y) \in A_{1j}, j = 0,2. \end{cases}$$

Then K^1 is piecewise continuous, and $\mathrm{div}(K^1) = 0$ on $\mathrm{int}(A_{ij})$, for $(i,j) \neq (1,1)$, but K^1 has a jump discontinuity on the line $x = \frac{1}{3}$, $x = \frac{2}{3}$. Thus, we have

$$K^1_x(\tfrac{1}{3}-, y) = J^x_{nx}(1, 3y), \quad K^1_x(\tfrac{1}{3}+, y) = \frac{3}{2}J^x_{nx}(1, 3y), \quad y \in [0, \tfrac{1}{3}].$$

We now modify K^1 to obtain a current satisfying the conditions of Proposition 2.2. Essentially, we use the current L, defined in Lemma 2.8, to move the excess current arriving at the left hand edges of the squares A_{10}, A_{12} to the right hand edge of A_{01}.

Let $L \in BV(F_{n-1}) \cap L^2(F_{n-1})$ be $L^{(n-1,m-1)}$ defined in Lemma 2.8. Recall that $L = 0$ except on $B^y_{m-1,0} \cap F_{n-1}$. Put

$$\begin{aligned} L^0(x,y) &= L(1-x, 1-y), \\ L^2(x,y) &= (L_x(1-x,y), -L_y(1-x,y)), \\ L^1(x,y) &= -L^0 - L^2. \end{aligned}$$

Since $\mathrm{div}(L) = 0$ on $B^y_{n-1,0} \cap F_{n-1}$, we have $\mathrm{div}(L^i) = 0$ for $0 \le i \le 2$. Define a vector field K^2 by

$$K^2(x,y) = \begin{cases} \frac{1}{2} L^j(3x, 3y-j), & (x,y) \in A_{0j}, 0 \le j \le 2, \\ 0, & (x,y) \in A_{1j}, j = 0, 2, \\ \frac{1}{2}(L^j_x(1-3x, 3y-j), -L^j_y(1-3x, 3y-j)), & (x,y) \in A_{2j}, 0 \le j \le 2. \end{cases}$$

Now let $K = K^1 + K^2$; then $K \in \mathcal{C}(F_n)$. To see this, note that for $0 \le y \le \frac{1}{3}$,

$$\begin{aligned} K^2_x(\tfrac{1}{3}-, y) &= \tfrac{1}{2} L^0_x(1-, 3y) = \tfrac{1}{2} L_x(0+, 1-3y) \\ &= \tfrac{1}{2} J^x_{nx}(0+, 1-3y) = \tfrac{1}{2} J^x_{nx}(1, 3y), \end{aligned}$$

so that

$$K^1_x(\tfrac{1}{3}-, y) + K^2_x(\tfrac{1}{3}-, y) = \frac{3}{2} J^x_{nx}(1, 3y) = K^1_x(\tfrac{1}{3}+, y).$$

With a number of similar calculations, this shows that $\mathrm{div}(K) = 0$. Therefore, using the symmetry of F_n and K,

(3.11) $R^x_n \le E(K,K) = 4E_{A_{00}}(K,K) + 2E_{A_{01}}(K,K) + 2E_{A_{10}}(K,K)$,

and it remains to estimate the terms in (3.11).

Note first that

$$E_{A_{10}}(K,K) = E_{A_{10}}(K^1,K^1) = \left(\frac{3}{2}\right)^2 \frac{1}{9} R^x_{n-1} = \frac{1}{4} R^x_{n-1},$$

and

$$E_{A_{01}}(K^1,K^1) = E_{A_{00}}(K^1,K^1) = \frac{1}{9} R^x_{n-1}.$$

Let $H = [0, \frac{1}{3} - 3^{-m}] \times [0,1]$, and $G = [\frac{1}{3} - 3^{-m}, \frac{1}{3}] \times [0,1]$. As $K^2 = 0$ on H we have for $j = 0, 1$,

$$\begin{aligned} E_{A_{0j}}(K,K) &= E_{A_{0j} \cap H}(K^1,K^1) + E_{A_{0j} \cap G}(K^1 + K^2, K^1 + K^2) \\ &\le E_{A_{0j}}(K^1,K^1) + E_{A_{0j} \cap G}(K^1,K^1) + 2E_{A_{0j}}(K^2,K^2). \end{aligned}$$

Using symmetry, and Lemma 2.7, for $j = 0, 1$

$$E_{A_{0j}\cap G}(K^1, K^1) = \frac{1}{9}E[F_{n-1} \cap B^y_{m-1}](J^x_{n-1}, J^x_{n-1}) \le \frac{1}{9}2^{-(m-1)}R^x_{n-1}.$$

From the definition of K^2,

$$E_{A_{00}}(K^2, K^2) = \frac{1}{36}E_{F_{n-1}}(L, L),$$

and

$$\begin{aligned} E_{A_{01}}(K^2, K^2) &= \frac{1}{36}E_{F_{n-1}}(L^1, L^1) \\ &\le \frac{1}{36}(2E_{F_{n-1}}(L^0, L^0) + 2E_{F_{n-1}}(L^2, L^2)) \\ &= \frac{1}{9}E_{F_{n-1}}(L, L). \end{aligned}$$

Finally, by Lemma 2.8,

$$E_{F_{n-1}}(L, L) \le 2^{-(m-1)}R^x_{n-1} + 3^{m-1}R^y_{n-m}.$$

Therefore, substituting in (3.11),

$$\begin{aligned} R^x_n &\le 6E_{A_{00}}(K^1, K^1) + 6E_{A_{00}\cap G}(K^1, K^1) + \frac{2}{3}E_{F_{n-1}}(L, L) + 2E_{A_{10}}(K^1, K^1) \\ &\le \frac{7}{6}R^x_{n-1} + \frac{8}{3}2^{-m}R^x_{n-1} + \frac{2}{3}(2^{-(m-1)}R^x_{n-1} + 3^{m-1}R^y_{n-m}) \\ &= \frac{7}{6}\left[(1 + \frac{16}{7}2^{-m})R^x_{n-1} + \frac{4}{21}3^m R^y_{n-m}\right], \end{aligned}$$

which completes the proof of Proposition 3.3. □

3.2 Proof of Theorem 1.1.

Fix $r > 0$. The left hand inequalities of Propositions 3.2 and 3.3 imply, for $n \ge k \ge 0$,

$$\text{(3.12)} \quad R^x_k(r)^{-1} \le \left(\frac{3}{2}\right)^{n-k} R^x_n(r)^{-1}, \text{ and } R^y_k(r) \le \left(\frac{6}{7}\right)^{n-k} R^y_n(r),$$

hence

$$\text{(3.13)} \qquad H_k(r) \le \frac{9}{7}H_{k+1}(r), \quad k \ge 0.$$

Since, by (3.12), we have $R^y_{n-m}(r) \le (6/7)^{m-1} R^y_{n-1}(r)$, for $n \ge m \ge 1$, it follows from (3.3) that for $n \ge m \ge 2$,

$$\begin{aligned} R^x_n(r) &\le \frac{7}{6}\left(1 + a_2 2^{-m} + A_2 3^m \frac{R^y_{n-m}(r)}{R^x_{n-1}(r)}\right) R^x_{n-1}(r) \\ &\le \frac{7}{6}\left(1 + a_2 2^{-m} + \frac{7}{6} A_2 {\theta_2}^m H_{n-1}(r)\right) R^x_{n-1}(r)\,, \end{aligned}$$

where $\theta_2 = 18/7$. Similarly, we have

$$R^y_n(r)^{-1} \le \frac{2}{3}\left(1 + a_1 2^{-m} + \frac{2}{3} A_1 {\theta_1}^m H_{n-1}(r)\right) R^y_{n-1}(r)^{-1}\,, \quad n \ge m \ge 2\,,$$

where $\theta_1 = 9/2$. Combining these inequalities, we obtain

$$H_n(r) \ge \frac{H_{n-1}(r)}{G_m(H_{n-1}(r))}\,, \quad n \ge m \ge 2\,, \tag{3.14}$$

where

$$G_m(x) = \frac{7}{9}(1 + a_1 2^{-m} + \frac{2}{3} A_1 \theta_1^m x)(1 + a_2 2^{-m} + \frac{7}{6} A_2 \theta_2^m x)\,. \tag{3.15}$$

Now let m be large enough so that $G_m(0) < 1$, and let $\delta_m > 0$ be such that $G_m(\delta_m) = 1$. Let η be an arbitrary number satisfying $0 \le \eta < \delta_m$, and put $\alpha = 1/G_m(\eta)$. We have $G_m(x)^{-1} \ge \alpha > 1$ for $0 \le x \le \eta$. Hence, by (3.14),

$$H_{n+1}(r) \ge \alpha\, H_n(r), \quad \text{whenever } H_n(r) \le \eta. \tag{3.16}$$

It follows immediately that there exists an integer $n_0 \ge m$ such that $H_{n_0}(r) > \eta$. Now if $k \ge n_0$ and $H_k(r) \ge \frac{7}{9}\eta$, then if $H_k(r) \le \eta$, by (3.16) $H_{k+1}(r) > H_k(r) \ge \frac{7}{9}\eta$. On the other hand, if $H_k(r) > \eta$ then by (3.13) $H_{k+1}(r) > \frac{7}{9}H_k(r) > \frac{7}{9}\eta$. Thus in either case $H_{k+1}(r) > \frac{7}{9}\eta$, and so, by induction, we deduce that

$$H_n(r) \ge \frac{7}{9}\eta, \quad \text{for } n \ge n_0.$$

This holds for any $\eta < \delta_m$, hence

$$\liminf_{n\to\infty} H_n(r) \ge \frac{7}{9}\delta_m\,.$$

Since this holds for any $r > 0$, and $H_n(r) = H_n(1/r)^{-1}$, we also deduce that

$$\limsup_{n\to\infty} H_n(r) \le \frac{9}{7}(\delta_m)^{-1}\,,$$

proving the Theorem. □

Remark. A numerical bound for the asymptotic values of $H_n(r)$ is obtained by computing δ_m. If we use the explicit values for the constants in G_m, we find $G_m(0) < 1$ for $m \geq 5$, and that $\delta_5 \geq 2.03039 \times 10^{-4}$, which leads to the numbers given in Section 1.

3.3 Proof of Theorem 1.3.

We begin with a lemma.

Lemma 3.5. *Let $f_n(r)$, $r \in [0, \infty)$, $n \geq 0$, be a sequence of functions satisfying, for constants $\alpha > 1$, $\beta > 0$, $\theta > 1$, $c_i \in (0, \infty)$,*

$$\beta f_{n-1}(r) \leq f_n(r) \leq \beta f_{n-1}(r)(1 + c_1 2^{-m} + r c_2 \alpha^m \theta^n), \tag{3.17}$$

for all $n \geq 1$ and $m \geq 2$. Then if $\xi = \log 2/\log(2\alpha)$ there exist constants s_0, c_5, depending only on α, θ, c_i such that

$$1 \leq \frac{\beta^{-n} f_n(\theta^{-n} s)}{f_0(\theta^{-n} s)} \leq \exp(c_5 s^\xi), \quad 0 < s \leq s_0, n \geq 1.$$

Proof. Let $n \geq 1$ be fixed, and choose $m_i \geq 2$ for $1 \leq i \leq n$. Then iterating (3.17) we obtain for $r > 0$

$$\beta^n \leq f_n(r)/f_0(r) \leq \beta^n \prod_{i=1}^{n} (1 + c_1 2^{-m_i} + r c_2 \alpha^{m_i} \theta^i).$$

So, setting $r = \theta^{-n} s$, $k_i = m_{n-i}$, $j = n - i$ we have

$$0 \leq \log(\beta^{-n} f_n(\theta^{-n} s)/f_0(\theta^{-n} s)) \leq c_1 \sum_{j=0}^{n-1} 2^{-k_j} + c_2 s \sum_{j=0}^{n-1} \alpha^{k_j} \theta^{-j}.$$

Choose $b > 0$ such that $2^{-b} < 1$ and $\alpha^b < \theta$ (so b depends only on α, θ), let

$$a = \frac{\log(1/s)}{\log(2\alpha)},$$

and let k_j satisfy

$$a + bj \leq k_j < 1 + a + bj, \quad 0 \leq j \leq n - 1.$$

Then $k_j \geq 2$ provided $s \leq s_0 = (2\alpha)^{-2}$. Thus

$$\begin{aligned} 0 \leq \log(\beta^{-n} f_n(\theta^{-n}s)/f_0(\theta^{-n}s)) &\leq c_1 \sum_{j=0}^{\infty} 2^{-a-bj} + c_2 s\alpha^{a+1} \sum_{j=0}^{\infty} (\alpha^b\theta^{-1})^j \\ &= c_3(\alpha,\theta)2^{-a} + c_4(\alpha,\theta)\alpha^a s \\ &= c_5(\alpha,\theta)s^{\xi}. \end{aligned}$$

□

Proof of Theorem 1.3.

The left hand side inequalities of Propositions 3.2 and 3.3 imply that, for $z = x, y$

(3.18) $$R_n^z(r)^{-1} \leq (6/7)^n R_0^z(r)^{-1}, \quad R_n^z(r) \leq (3/2)^n R_0^z(r).$$

It follows that (treating the cases $m \leq n$, $m > n$ separately)

$$\frac{R_{n-m}^y(r)}{R_{n-1}^x(r)} \leq (7/6)(9/7)^n r, \quad n \geq 1, m \geq 1.$$

Therefore (3.3) implies that for $n \geq 1$, $m \geq 2$,

$$\frac{7}{6}R_{n-1}^x(r) \leq R_n^x(r) \leq \frac{7}{6}R_{n-1}^x(r)(1 + a_2 2^{-m} + r(7/6)A_2(9/7)^n 3^m).$$

So, by Lemma 3.5, taking $f_n(r) = R_n^x(r)$, $\beta = 7/6$, $\theta = 9/7$, $\alpha = 3$, $\xi_1 = \log 2/\log 6$, we obtain

(3.19) $$1 \leq (6/7)^n R_n^x((7/9)^n s) \leq \exp(cs^{\xi_1}), \quad n \geq 1, s \leq s_0.$$

Here $s_0 = 1/36$ and $c \in (0, \infty)$.

Using Lemma 3.1 and (3.19), we obtain, replacing s by s^{-1},

(3.20) $$1 \leq (2/3)^n R_n^y((9/7)^n s)s^{-1} \leq \exp(cs^{-\xi_1}), \quad n \geq 1, s \geq s_0^{-1}.$$

In a similar fashion we have, if $n \geq 1$, $m \geq 1$, $k = \max(n-m, 0)$,

$$\frac{R_{n-1}^x(r)}{R_{n-m}^y(r)} \leq (3/2)^{n-1}(6/7)^k r^{-1} \leq (2/(3r))(9/7)^n(7/6)^m.$$

So, using (3.1), and replacing r by r^{-1}, for $n \geq 1$, $m \geq 2$,

(3.21) $$\begin{aligned} &\tfrac{2}{3}R_{n-1}^x(1/r)^{-1} \leq R_n^x(1/r)^{-1} \\ &\leq \tfrac{2}{3}R_{n-1}^x(1/r)^{-1}(1 + a_1 2^{-m} + r(2A_1/3)(9/7)^n(7/2)^m). \end{aligned}$$

Taking $f_n(r) = R_n^x(1/r)^{-1}$, $\beta = 2/3$, $\theta = 9/7$, $\alpha = 7/2$, $\xi_2 = \log 2/\log 7$, we obtain by Lemma 3.5,

$$(3.22) \qquad 1 \leq (3/2)^n R_n^x((9/7)^n s)^{-1} \leq \exp(cs^{-\xi_2}), \quad n \geq 1, s \geq s_0^{-1}.$$

Using Lemma 3.1 this implies that

$$(3.23) \qquad 1 \leq (7/6)^n R_n^y((7/9)^n s)^{-1} s \leq \exp(cs^{\xi_2}), \quad n \geq 1, s \leq s_0.$$

Multiplying together (3.22) and (3.20) gives the theorem. □

4 Asymptotic behavior of effective resistances.

4.1 Statement of the results.

For the isotropic case $r = 1$, it is proved in [2] that there exists a constant $\rho > 1$ such that

$$(4.1) \qquad 4^{-1}\rho^n \leq R_n \leq 4\,\rho^n\,, \quad n \geq 0,$$

where $R_n \stackrel{\text{def}}{=} R_n^x(1) = R_n^y(1)$. (It is also proved there that $7/6 \leq \rho \leq 1.27656$; calculations of R_n, $1 \leq n \leq 7$ suggest that $\rho \approx 1.25149$. See [2] and [4]). The proof uses the inequalities

$$4^{-1} R_m\, R_n \leq R_{n+m} \leq 4\, R_m\, R_n\,, \quad n \geq 0,\ m \geq 0.$$

The following proposition extends this result to the anisotropic case $r \neq 1$. Theorem 1.2 follows at once if we put $m = 0$ in Proposition 4.1.

Proposition 4.1. *For $z = x, y$, $r > 0$, $n, m \in \mathbb{Z}_+$,*

$$(4.2) \qquad R_{n+m}^z(r)^{-1} \;\leq\; 16\,\rho^{-n}\,(R_m^x(r)^{-1} + R_m^y(r)^{-1})\,,$$

$$(4.3) \qquad R_{n+m}^z(r) \;\leq\; 8\,\rho^n\,(R_m^x(r) + R_m^y(r)).$$

Remark. The proof below also implies the bounds with R_n in place of ρ^n in both (4.2) and (4.3).

To prove the Proposition, we first recall results proved in [2], which relate R_n to the effective resistances for crosswire resistance networks. For $i = 0, 1, \cdots, 3^n - 1$, and $j = 0, 1, \cdots, 3^n - 1$, let Λ_{ij} be the closure in $\mathbb{R}^2$ of

$$([i3^{-n},\,(i+1)3^{-n}] \times [j3^{-n},\,(j+1)3^{-n}]) \cap F_n\,,$$

and let

$$S_n = \{(i,j) \in \{0,1,\cdots,3^n-1\}^2 : A_{ij} \neq \emptyset\}\,.$$

Given $a = \{a_{i,j} : i = 0,1,\cdots,3^n,\ j = 0,1,\cdots,3^n\}$, set

$$\bar{a}_{ij} = 4^{-1}\sum_{\alpha=0}^{1}\sum_{\beta=0}^{1} a_{i+\alpha,j+\beta}. \tag{4.4}$$

and define

$$K^D(a) = \sum_{(i,j)\in S}\sum_{\alpha=0}^{1}\sum_{\beta=0}^{1}(a_{i+\alpha,j+\beta} - \bar{a}_{ij})^2\,.$$

Define R_n^D by

$$(R_n^D)^{-1} = \inf_a\{K^D(a) \mid a_{0,j} = 0,\ a_{3^n,j} = 1,\quad j = 0,1,\cdots,3^n\}\,. \tag{4.5}$$

The notation R_n^D is consistent with that of [2], and denotes the effective resistance of the wire network obtained by replacing each board of side 3^{-n} in F_n by a diagonal crosswire of 4 unit resistors. Next let Λ_{ij} and S_n be as above. Assume that a set of numbers

$$J = \{J_{ij\eta} \mid i = 0,1,\cdots,3^n-1,\ j = 0,1,\cdots,3^n-1,\ \eta = 1,2,3,4\}$$

satisfies the following conditions:

$$\begin{cases} J_{ij\eta} = 0,\ (i,j) \in \{0,1,\cdots,3^n-1\}^2 \setminus S,\ \eta = 1,2,3,4, \\ \displaystyle\sum_{\eta=1}^{4} J_{ij\eta} = 0,\ (i,j) \in S, \\ J_{i+1,j,1} + J_{ij3} = 0,\ (i,j) \in \{0,1,\cdots,3^n-1\}^2\,, \\ J_{i,j+1,2} + J_{ij4} = 0,\ (i,j) \in \{0,1,\cdots,3^n-1\}^2\,. \end{cases} \tag{4.6}$$

We regard $J_{ij\eta}$ as being the current flowing in the wire network G_n obtained by replacing each board of side 3^{-n} in F_n by a horizontal and vertical crosswire of 4 wires, each of resistance $\frac{1}{2}$. With this interpretation (4.6) are the equations of current conservation.

We impose the following 'boundary conditions':

$$\begin{cases} J_{i02} = J_{i,3^n-1,4} = 0,\ i = 0,1,\cdots,3^n-1, \\ \displaystyle\sum_{j=0}^{3^n-1} J_{0j1} = -\sum_{j=0}^{3^n-1} J_{3^n-1,j,3} = 1. \end{cases} \tag{4.7}$$

Put

$$K^G(J) = \frac{1}{2} \sum_{(i,j)\in S} \sum_{\eta=1}^{4} J_{ij\eta}^2 \,,$$

and

$$R_n^G = \inf_J \{ K^G(J) \mid J \text{ satisfies (4.6) and (4.7).} \} \,.$$

The notation R_n^G is consistent with that of [2], and denotes the effective resistance of the network G_n.

From [2] (see Theorem 3.3, Proposition 4.1, Theorem 4.3 and (5.4)) we have

Lemma 4.2. *For* $n \geq 0$,

$$R_n^G \leq 4 \min(\rho^n, R_n) \leq 4 \max(\rho^n, R_n) \leq 8 R_n^D \,. \tag{4.8}$$

4.2 Proof of Proposition 4.1.

It is sufficient to consider the case $z = x$, as the case $z = y$ then follows immediately by Lemma 3.1. We first prove (4.2). For $i = 0, 1, \cdots, 3^n - 1$, and $j = 0, 1, \cdots, 3^n - 1$, let B_{ij} be the closure in $\mathbb{R}^2$ of $([i3^{-n},\ (i+1)3^{-n}] \times [j3^{-n},\ (j+1)3^{-n}]) \cap F_{n+m}$. Then $\bar{F}_{n+m} = \bigcup_{i,j} B_{ij}$, and each non-empty B_{ij} is congruent to $3^{-n}\bar{F}_m$. Define four functions $\varphi_{\alpha\beta}$, $\alpha = 0, 1$, $\beta = 0, 1$, on $\bar{F}_m$ by

$$\begin{aligned}
\varphi_{11}(x,y) &= V_m^x(r)(x,y)\, V_m^y(r)(x,y) \,, \\
\varphi_{01}(x,y) &= (1 - V_m^x(r)(x,y))\, V_m^y(r)(x,y) \,, \\
\varphi_{10}(x,y) &= V_m^x(r)(x,y)\, (1 - V_m^y(r)(x,y)) \,, \\
\varphi_{00}(x,y) &= (1 - V_m^x(r)(x,y))\, (1 - V_m^y(r)(x,y)) \,.
\end{aligned}$$

Note that Lemma 2.4, (2.3), and (2.1) imply

$$\mathcal{E}_{F_m}(\varphi_{\alpha\beta}, \varphi_{\alpha\beta}) \leq 2\, (R_m^x(r)^{-1} + R_m^y(r)^{-1}) \,. \tag{4.9}$$

Given a set of real numbers $\{a_{i,j} \mid 0 \leq i, j \leq 3^n\}$, with $\bar{a}_{ij}$ defined by (4.4), define $v \in C(\bar{F}_{n+m}) \cap H^1(F_{n+m})$ by:

$$v(x,y) \stackrel{\text{def}}{=} \bar{a}_{ij} + \sum_{\alpha=0}^{1} \sum_{\beta=0}^{1} (a_{i+\alpha,j+\beta} - \bar{a}_{ij})\, \varphi_{\alpha\beta}(3^n x - i, 3^n y - j) \,,$$
$$(x,y) \in B_{ij}, \quad (i,j) \in S_n \,.$$

Note that if (a_{ij}) satisfy the 'boundary conditions' in (4.5) then v satisfies (1.3). Continuity of v at the boundaries of the B_{ij} follows from (2.4). Recalling that B_{ij} is congruent to $3^{-n}\bar{F}_m$ for $(i,j) \in S$, we have

$$\begin{aligned}
\mathcal{E}_{F_{n+m}}(v,v) &= \sum_{(i,j)\in S} \mathcal{E}_{B_{ij}}(v,v) \\
&= \sum_{(i,j)\in S} \sum_{\alpha,\beta,\alpha',\beta'} (a_{i+\alpha,j+\beta} - \bar{a}_{ij})(a_{i+\alpha',j+\beta'} - \bar{a}_{ij})\mathcal{E}_{F_m}(\varphi_{\alpha\beta}, \varphi_{\alpha'\beta'}) \\
&\le \sum_{(i,j)\in S} \sum_{\alpha,\beta,\alpha',\beta'} \frac{1}{2}\Big\{(a_{i+\alpha,j+\beta} - \bar{a}_{ij})^2 \mathcal{E}_{F_m}(\varphi_{\alpha\beta}, \varphi_{\alpha\beta}) \\
&\qquad + (a_{i+\alpha',j+\beta'} - \bar{a}_{ij})^2 \mathcal{E}_{F_m}(\varphi_{\alpha'\beta'}, \varphi_{\alpha'\beta'})\Big\} \\
&= 4 \sum_{(i,j)\in S} \sum_{\alpha,\beta} (a_{i+\alpha,j+\beta} - \bar{a}_{ij})^2 \mathcal{E}_{F_m}(\varphi_{\alpha\beta}, \varphi_{\alpha\beta}) \\
&\le 8 \sum_{(i,j)\in S} \sum_{\alpha,\beta} (a_{i+\alpha,j+\beta} - \bar{a}_{ij})^2 (R_m^x(r)^{-1} + R_m^y(r)^{-1}) \\
&\le 8K^D(a)(R_m^x(r)^{-1} + R_m^y(r)^{-1}),
\end{aligned}$$

where we used (4.9) in the last line. Hence, taking infimum over $\{a_{ij}\}$ and using (4.5) we have

$$R_{n+m}^x(r)^{-1} \le 8(R_n^D)^{-1}\,(R_m^x(r)^{-1} + R_m^y(r)^{-1}),$$

and (4.2) now follows immediately using (4.8).

We now turn to a proof of (4.3). Let B_{ij} and S_n be as above. Define currents $I_{\eta\eta'}$, $1 \le \eta, \eta' \le 4$, on $\bar{F}_m$ as follows. First, let

$$I_{13} = -I_{31} = J_m^x(r), \; I_{24} = -I_{42} = J_m^y(r).$$

Let $I_{12} = (I_{12x}, I_{12y})$ be the current $L^{(m,0)}$ defined in Lemma 2.8, and let

$$\begin{aligned}
&I_{14}(x,y) = -I_{41}(x,y) = (I_{12x}(x,1-y), -I_{12y}(x,1-y)), \\
&I_{32}(x,y) = -I_{23}(x,y) = (-I_{12x}(1-x,y), I_{12y}(1-x,y)), \\
&I_{43}(x,y) = -I_{34}(x,y) = (I_{12x}(1-x,1-y), I_{12y}(1-x,1-y)), \quad (x,y) \in \bar{F}_m.
\end{aligned}$$

Finally we put $I_{\eta\eta} = 0$, $\eta = 1,2,3,4$. From Lemma 2.8 we have,

$$E_{F_m}(I_{\eta\eta'}, I_{\eta\eta'}) \le R_m^x(r) + R_m^y(r), \quad \eta, \eta' \in \{1,2,3,4\}. \tag{4.10}$$

Note also that from (2.10) we have the boundary conditions

$$\begin{aligned}
I_{1\eta,x}(0,y) &= -I_{\eta 1,x}(0,y) = J_{mx}^x(r)(0,y), & \eta &= 2,3,4, \\
I_{2\eta,y}(x,0) &= -I_{\eta 2,y}(x,0) = J_{my}^y(r)(x,0), & \eta &= 1,3,4, \\
I_{3\eta,x}(1,y) &= -I_{\eta 3,x}(1,y) = -J_{mx}^x(r)(1,y), & \eta &= 1,2,4, \\
I_{4\eta,y}(x,1) &= -I_{\eta 4,y}(x,1) = -J_{my}^y(r)(x,1), & \eta &= 1,2,3,
\end{aligned}$$

while for the remaining combinations of the suffices, the corresponding quantities vanish.

Given $\{J_{ij\eta}\}$ satisfying (4.6), write $J^{\pm}_{ij\eta} = 2^{-1}(|J_{ij\eta}| \pm J_{ij\eta}))$,

$$h_{ij} = \sum_{\eta=1}^{4} J^{+}_{ij\eta} = \sum_{\eta=1}^{4} J^{-}_{ij\eta}\,,$$

and define a current I on F_{n+m}, by

$$I(x,y) \stackrel{\text{def}}{=} \frac{1}{h_{ij}} \sum_{\eta=0}^{4} \sum_{\eta'=0}^{4} J^{+}_{ij\eta} J^{-}_{ij\eta'} I_{\eta\eta'}(3^n x - i, 3^n y - j)\,,$$
$$(x,y) \in B_{ij}, \quad (i,j) \in S.$$

Then $I \in \mathcal{C}(F_{n+m})$, so if $\{J_{ij\eta}\}$ satisfy (4.7), then by (2.6) we have

(4.11) $$R^x_{n+m}(r) \leq E_{F_{n+m}}(I,I).$$

Recalling that $B_{ij} \cong 3^{-n}\bar{F}_m$, $(i,j) \in S$, we have

$$\begin{aligned}
&E_{F_{n+m}}(I,I)\\
&= \sum_{(i,j)\in S} E_{B_{ij}}(I,I)\\
&= \sum_{(i,j)\in S} h_{ij}^{-2} \sum_{\eta\eta'} \sum_{\xi\xi'} J^{+}_{ij\eta} J^{-}_{ij\eta'} J^{+}_{ij\xi} J^{-}_{ij\xi'} E(I_{\eta\eta'}, I_{\xi\xi'})\\
&\leq 2^{-1} \sum_{(i,j)\in S} h_{ij}^{-2} \sum_{\eta\eta'} \sum_{\xi\xi'} J^{+}_{ij\eta} J^{-}_{ij\eta'} J^{+}_{ij\xi} J^{-}_{ij\xi'} (E_{F_m}(I_{\eta\eta'}, I_{\eta\eta'}) + E_{F_m}(I_{\xi\xi'}, I_{\xi\xi'}))\\
&\leq (R^x_m(r) + R^y_m(r)) \sum_{(i,j)\in S} h_{ij}^2\,,
\end{aligned}$$

where we used (4.10) in the last line.

Now by the Cauchy-Schwarz inequality,

$$h_{ij} = \tfrac{1}{2} \sum_{\eta} |J_{ij\eta}| \leq (\sum_{\eta} J^2_{ij\eta})^{1/2}.$$

Hence

$$\begin{aligned}
R^x_{n+m}(r) &\leq E_{F_{n+m}}(I,I)\\
&\leq (R^x_m(r) + R^y_m(r)) \sum_{ij} \sum_{\eta} J^2_{ij\eta}\\
&\leq 2(R^x_m(r) + R^y_m(r)) K^G(J)\\
&\leq 2(R^x_m(r) + R^y_m(r)) R^G_n\,,
\end{aligned}$$

and using (4.8) gives (4.3). $\square$

A Proof of Lemma 2.8.

In this Appendix, we will give a proof of Lemma 2.8. In fact we prove a more general result:

Lemma A.1. *Let $0 < k_x \leq 1$ and $0 \leq k_y < 1$, and let $B = ((0, k_x) \times (k_y, 1)) \cap F_n$ and $\tilde{B} = ((0, k_x) \times (0, 1)) \cap F_n$. Let $v_0 < v_1$, $v_0' < v_1'$ be constants, and let v^x be the harmonic function on B, with Neumann boundary conditions $\frac{\partial v^x}{\partial n} = 0$ at the boundaries (in $\mathbb{R}^2$) of B, except at $x = 0$ and $x = k_x$, where the Dirichlet boundary conditions, $v^x(0, y) = v_1$ and $v^x(k_x, y) = v_0$ are imposed. Define a current $j^x = (j^x_x, j^x_y)$ on B by $j^x = R^x \nabla v^x$, where the constant R^x is defined by the normalization condition $\int_{k_y}^1 j^x_x(0, y)\, dy = 1$. Similarly, let v^y be the harmonic function on $\tilde{B}$, with Neumann boundary conditions, except at $y = 1$ and $y = 0$, where Dirichlet boundary conditions $v^y(x, 1) = v_0'$ and $v^y(x, 0) = v_1'$ are imposed. Define $j^y = (j^y_x, j^y_y) = -R^y \nabla v^y$, where R^y is defined by $\int_0^{k_x} j^y_y(x, 1)\, dx = 1$. Then, there exist two disjoint open subsets of B, B^x and B^y, satisfying the following:*

1. *the boundary of B^x contains $(\{0\} \times [k_y, 1]) \cup (([0, k_x] \times \{k_y\}) \cap \partial B)$, and has no common points with $((0, k_x] \times \{1\}) \cup ((\{k_x\} \times (k_y, 1]) \cap \partial B)$,*
2. *the boundary of B^y contains $([0, k_x] \times \{1\}) \cup ((\{k_x\} \times [k_y, 1]) \cap \partial B)$, and has no common points with $(\{0\} \times [k_y, 1)) \cup (([0, k_x) \times \{k_y\}) \cap \partial B)$,*
3. *The vector field J defined by*

(A.1)
$$J(x, y) = \begin{cases} j^x(x, y)\,, & (x, y) \in B^x, \\ j^y(x, y)\,, & (x, y) \in B^y, \\ 0, & \text{otherwise}, \end{cases}$$

is in $\mathcal{C}(B)$.

It follows, in particular, that

(A.2)
$$E_B(J, J) \leq E_B(j^x, j^x) + E_B(j^y, j^y)\,.$$

Proof. Note that with a linear transformation of the coordinate $y' = y\sqrt{r}$, (2.2) becomes the Laplace equation in the standard sense. Hence, the potential functions V_n^x and V_n^y are harmonic functions in the usual sense, with this change of coordinate. We assume this change of coordinate in the following. With the change of coordinate, the domain F_n may no more be a square, but it is still a rectangle shaped object, with rectangular 'holes' inside. To avoid the clumsiness in the notation, we will keep the notations F_n and assume that it is a square $[0,1]^2$ with square holes inside. We will not use any symmetries specific to squares in the proofs, and the results are directly applicable to the original problem.

Put $v = R^x v^x - R^y v^y$. Since v is harmonic, there locally exists, around each point in B, an analytic function $u(x+y\sqrt{-1}) \stackrel{\text{def}}{=} v(x,y) + \sqrt{-1}\,w(x,y)$, where w is the conjugate harmonic function of v.

Note that for any closed path C in B, we have

$$\oint_C \text{grad} w \cdot dx = \oint_C \frac{\partial v}{\partial n} ds = -\oint_{\partial' B} \frac{\partial v}{\partial n} ds,$$

where n is the unit normal vector and $\partial' B$ is the boundary of B in the interior of C. In the last equality, we used the fact that v is harmonic. Because of the boundary conditions on v we see that this quantity is zero, hence w is single valued.

Denote the boundary of B by ∂B. (By a boundary of a set, we always mean, in the following, that as a set in $\mathbb{R}^2$.) Decompose ∂B into the 'external' boundary of B defined by

$$\partial_{ext} B = \partial B \cap (\{x=0\} \cup \{x = k_x\} \cup \{y = k_y\} \cup \{y = 1\}),$$

and the 'internal' boundary defined by $\partial_{int} B = \partial B \setminus \partial_{ext} B$. Decompose $\partial_{int} B \setminus \partial_{corner} B = \partial_{int}^o B$, where $\partial_{corner} B$ is the (finite) set of corner points of square 'holes' in $B \subset F_n$. By the reflection principle of analytic functions (see the arguments in the Appendix of [2] for F_n with boundary conditions dealt with here), we see that $u(z)$ can be analytically continued to a neighborhood of each point in $\partial B \setminus \partial_{corner} B = \partial_{ext} B \cup \partial_{int}^o B$, and that at each $z_0 \in \partial_{corner} B$, there exists an analytic function U in a neighborhood of 0, such that $u(z) = U((z - z_0)^{2/3})$. We regard, in the following, u (and also v, w) as a continuous function on the closed set $\bar{B}$, analytic on $\bar{B} \setminus \partial_{corner} B$.

Note also that similar considerations hold for v^x on B and v^y on $\tilde{B} \supset B$ in place of v. We define w^x on B and w^y on $\tilde{B}$ which are conjugate harmonic functions of v^x and v^y, respectively, and put

$$u^x(x + \sqrt{-1}y) = v^x(x,y) + \sqrt{-1}\,w^x(x,y)$$

and

$$u^y(x+\sqrt{-1}y) = v^y(x,y) + \sqrt{-1}\, w^y(x,y)\,.$$

Obviously, we can fix constant ambiguities of conjugate harmonic functions to satisfy $w = R^x w^x - R^y w^y$ and $u = R^x u^x - R^y u^y$.

Decompose $\partial_{ext}B$ into 4 parts and put $e_1 = \partial_{ext}B \cap \{x = 0\}$, $e_2 = \partial_{ext}B \cap \{y = k_y\}$, $e_3 = \partial_{ext}B \cap \{x = k_x\}$, $e_4 = \partial_{ext}B \cap \{y = 1\}$.

Define 2 disjoint open subsets B^x, B^y of B by $B^x \stackrel{\text{def}}{=} \{(x,y) \in B \mid w(x,y) > w(0,1)\}$ and $B^y \stackrel{\text{def}}{=} \{(x,y) \in B \mid w(x,y) < w(0,1)\}$. We will prove that B^x and B^y satisfy the statements of the Lemma.

A point $(x,y) \in B \setminus \partial_{corner}B$ is said to be a critical point of v^x if $\nabla v^x(x,y) = 0$, or equivalently, $u^{x\prime}(z) = 0$. By the uniqueness theorem on analytic continuation, we see that there are at most a finite number of critical points in $B \setminus \partial_{corner}B$. (The possibility of accumulation of critical points to a point in $\partial_{corner}B$ is ruled out by considering the uniqueness theorem on U^x, the function corresponding to U defined above.) Denote the (finite) set of critical points by A^x_{crit}.

Note first that, by assumption, we have $v^x(x,y) \le v_1$, from which follows $\dfrac{\partial v^x}{\partial x}(0,y) \le 0$. The boundary condition $v^x(0,y) = v_1$ implies $\dfrac{\partial v^x}{\partial y}(0,y) = 0$. These results and the fact that the number of critical points is finite imply, with the Cauchy–Riemann relation, that $\dfrac{\partial w^x}{\partial y}(0,y) < 0$ except for at most finite number of ys'. Note also that we have, for v^y, the boundary condition $\dfrac{\partial v^y}{\partial x}(0,y) = 0$. With the Cauchy–Riemann relation we have $\dfrac{\partial w^y}{\partial y}(0,y) = 0$. Therefore we have $\dfrac{\partial w}{\partial y}(0,y) = R^x \dfrac{\partial w^x}{\partial y}(0,y) - R^y \dfrac{\partial w^y}{\partial y}(0,y) < 0$, except for at most finite number of points on e_1, consequently, $w(0,y) > w(0,1)$ for $k_y \le y < 1$. w is a continuous function on $\bar{B}$, hence we see that e_1 is contained in the boundary of B^x and has no common points with that of B^y except for a point $(0,1)$. The positivity of R^x and R^y are consequences of the fact that the assumptions imply that R^x and j^x (or R^y and j^y) satisfy a relation analogous to (2.6) and (2.7).

Similarly we deduce that e_4 is contained in the boundary of B^y and has no common points with that of B^x except for a point $(0,1)$.

To prove that e_3 is contained in the boundary of B^y, first note that, by the Cauchy–Riemann relations and the boundary conditions on v^x and v^y

and the normalization condition on j^y, we have

$$\begin{aligned} w(k_x,1) - w(0,1) &= \int_0^{k_x} \frac{\partial w}{\partial x}(x,1)\,dx \\ &= \int_0^{k_x} -R^x \frac{\partial v^x}{\partial y}(x,1) + R^y \frac{\partial v^y}{\partial y}(x,1)\,dx = -\int_0^{k_x} j_y^y(x,1)\,dx = -1\,. \end{aligned}$$

Then we have, using Cauchy–Riemann relations and the assumption that $\dfrac{\partial v^y}{\partial x} = 0$ on e_3,

$$\begin{aligned} w(k_x,y) - w(0,1) &= w(k_x,y) - w(k_x,1) - 1 \\ &= -\int_y^1 \frac{\partial v}{\partial x}(k_x,y)\,dy - 1 = \int_y^1 j_x^x(k_x,y)\,dy - 1\,. \end{aligned}$$

Using $\mathrm{div} j^x = 0$, the Gauss–Green formula, and the boundary conditions on v^x, we see that $\int_{k_y}^1 j_x^x(k_x,y)\,dy = \int_{k_y}^1 j_x^x(0,y)\,dy = 1$. Therefore,

$$w(k_x,y) - w(0,1) = -\int_{k_y}^y j_x^x(k_x,y)\,dy\,. \tag{A.3}$$

By assumption, $v^x(x,y) \geq v_0$, from which follows $\dfrac{\partial v^x}{\partial x}(k_x,y) \leq 0$. The boundary condition $v^x(k_x,y) = v_0$ implies $\dfrac{\partial v^x}{\partial y}(k_x,y) = 0$. These results and the fact that the number of critical points is finite imply, $j_x^x(k_x,y) = -R^x \dfrac{\partial v^x}{\partial x}(k_x,y) > 0$, except for at most finite number of points. With (A.3) we see that $w(k_x,y) < w(0,1)$ on $e_3 \setminus (k_x,k_y)$, implying that e_3 is contained in the boundary of B^y, and has no common points with that of B^x except for (k_x,k_y).

To prove that e_2 is contained in the boundary of B^x, (and has no common points with that of B^y except for (k_x,k_y),) it suffices to prove $w(x,k_y) > w(0,1)$ on $e_2 \setminus (k_x,k_y)$. By an analogous argument to those above we obtain

$$w(0,k_y) - w(0,1) = -\int_{k_y}^1 \frac{\partial w}{\partial y}(0,y)\,dy = \int_{k_y}^1 j_x^x(0,y)\,dy = 1\,.$$

Noting the boundary condition $\dfrac{\partial v^x}{\partial y}(x,k_y) = 0$ on e_2, we therefore see, with the Cauchy–Riemann relations, that

$$w(x,k_y) - w(0,1) = \int_0^x \frac{\partial w}{\partial x}(x,k_y)\,dx + 1 = -\int_0^x j_y^y(x,k_y)\,dx + 1\,.$$

Hence $w(x,k_y) > w(0,1)$ on $e_2 \setminus (k_x,k_y)$ holds if we can show

$$\int_0^x j_y^y(x,k_y)\,dx < 1, \quad 0 \le x < k_x\,. \tag{A.4}$$

Suppose $\int_0^{x_0} j_y^y(x,k_y)\,dx \ge 1$ for some $(x_0,k_y) \in e_2 \setminus (k_x,k_y)$, and put

$$\ell^y = \{(x,y) \in \tilde{B} \mid w^y(x,y) = w^x(x_0,k_y)\}\,.$$

Since w^y is a harmonic function on $\tilde{B}$, ℓ^y is a smooth curve (or a set of smooth curves) in $\tilde{B}$, whose tangent is proportional to ∇v^y, which implies that v^y is strictly monotone on ℓ^y, hence it is not a closed orbit, and separates $\tilde{B}$ in domains with $w^y(x,y) > w^x(x_0,k_y)$ and with $w^y(x,y) < w^x(x_0,k_y)$. The boundary conditions $\dfrac{\partial v^y}{\partial x} = 0$ on the edges $x = 0$ and $x = k_x$ imply that w^y is constant on these edges, so that ℓ^y cannot have endpoints on them. Therefore there is an endpoint of ℓ^y on the edges $\{(x,0) \mid 0 \le x < k_x\} \cup \{(x,1) \mid 0 \le x < k_x\}$. Let $(x_1,1)$ be an endpoint of ℓ^y, satisfying $0 \le x_1 < k_x$. (The case that the endpoints are only on the edge $y = 0$ can be handled similarly.) There is a connected piecewise smooth curve $\ell^{y\prime} \subset \ell^y \cup \partial^o_{int} B$ which connects (x_0,k_y) to $(x_1,1)$. Consider the subset of B bounded by $\ell^{y\prime}$, e_2, e_1, and e_4. Applying the Gauss–Green formula and the current conservation $\mathrm{div} j^y = 0$, and noting that $j^y \cdot n = 0$ on $\ell^{y\prime}$ and e_1, where n is a normal vector, we see that

$$\int_0^{x_1} j_y^y(x,1)\,dx = \int_0^{x_0} j_y^y(x,k_y)\,dx \ge 1. \tag{A.5}$$

On the other hand, $v^y(x,1) = v_0' \le v^y(x,y)$ implies $\dfrac{\partial v^y}{\partial y}(x,1) \ge 0$, and $\dfrac{\partial v^y}{\partial x}(x,1) = 0$. This implies (with an argument similar to one which led to $w(0,y) > w(0,1)$ for $k_y \le y < 1$) that $j_y^y(x,1) > 0$, $0 \le x \le k_x$, except for at most finite number of points. Hence

$$\int_0^{x_1} j_y^y(x,1)\,dx = 1 - \int_{x_1}^{k_x} j_y^y(x,1)\,dx < 1\,.$$

This contradicts (A.5). Hence (A.4) is proved.

We are left with the statements on J defined in (A.1). Since j^x and j^y are in $\mathcal{C}(B)$, it follows at once that J is square integrable and of bounded variation. To prove that $\mathrm{div} J = 0$, let f be an infinitely differentiable function on B with compact support. Using (A.1), $\mathrm{div} j^x = \mathrm{div} j^y = 0$, and the Gauss–Green formula [12, p.340] in turn, we have

$$\begin{aligned}\int_B f\,\mathrm{div}J\,dx\,dy &= -\int_B (\nabla f)\cdot J\,dx\,dy \\ &= -\int_{B^x} (\nabla f)\cdot j^x\,dx\,dy - \int_{B^y} (\nabla f)\cdot j^y\,dx\,dy \\ &= -\int_{B^x} \mathrm{div}(f\,j^x)\,dx\,dy - \int_{B^y} \mathrm{div}(f\,j^y)\,dx\,dy \\ &= -\int_{\partial B^x} f\,j^x\cdot n\,ds - \int_{\partial B^y} f\,j^y\cdot n\,ds\,,\end{aligned}$$

where n is the unit normal vector to the curves ∂B^x or ∂B^y, in the outward directions of the domain B^x or B^y. Since f has compact support on B, the contribution to the line integration from ∂B is zero. On the other hand, the function w is analytic in B, hence,

$$\ell \stackrel{\mathrm{def}}{=} (\partial B^x)\setminus(\partial B) = (\partial B^y)\setminus(\partial B)\,,$$

and $w(x,y) = w(0,1)$ on the curve ℓ. Note that $\nabla v = -R^y\nabla v^y + R^x\nabla v^x = j^y - j^x$. By Cauchy–Riemann relations we know that $\nabla v\cdot\nabla w = 0$. Hence we have $(j^y - j^x)\cdot n = 0$ on ℓ, where n is the unit normal vector to ℓ, with same sign as n for ∂B^x. The normal vector n has opposite signs on ∂B^x and ∂B^y. Therefore,

$$\int_B f\,\mathrm{div}J\,dx\,dy = -\int_\ell f\,(j^x - j^y)\cdot n\,ds = 0\,,$$

which proves $\mathrm{div}J = 0$.

The estimate (A.2) now follows since

$$E_{F_n}(J,J) = E_{B^x\cap F_n}(J,J) + E_{B^y\cap F_n}(J,J) \le E_{F_n}(j^x,j^x) + E_{F_n}(j^y,j^y)\,.$$

□

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