Existence of an infinite particle limit of stochastic ranking process

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ABSTRACT

We study a stochastic particle system which models the time evolution of the ranking of books by online bookstores (e.g., Amazon.co.jp). In this system, particles are lined in a queue. Each particle jumps at random jump times to the top of the queue, and otherwise stays in the queue, being pushed toward the tail every time another particle jumps to the top. In an infinite particle limit, the random motion of each particle between its jumps converges to a deterministic trajectory. (This trajectory is actually observed in the ranking data on web sites.) We prove that the (random) empirical distribution of this particle system converges to a deterministic space-time dependent distribution. A core of the proof is the law of large numbers for *dependent* random variables.

Key words: stochastic ranking process; hydrodynamic limit; dependent random variables; law of large numbers

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1 Introduction.

1.1 Definitions.

Let $(\Omega, \mathcal{B}, \mathrm{P})$ be a probability space, and on this probability space we consider a stochastic ranking process $\{X_i^{(N)}(t) \mid t \geqq 0,\ i = 1, 2, \cdots, N\}$ of N particles, where $X_i^{(N)}(t)$ is the ranking of particle i at time t, defined as follows.

For each i assume that $x_{i,0}^{(N)} \in \{1, 2, \cdots, N\}$ and $w_i^{(N)} > 0$ are given. $x_{i,0}^{(N)}$ is the initial value of the ranking of the particle i; $x_{i,0}^{(N)} = X_i^{(N)}(0)$. We require that $x_{i,0}^{(N)}$, $i = 1, 2, \cdots, N$, are different numbers, or in other words, $x_{i,0}^{(N)}$, $i = 1, 2, \cdots, N$, is a permutation of $1, 2, \cdots, N$. $w_i^{(N)}$ is the jump rate of the particle i.

For each i let $\tau_{i,j}^{(N)}$, $j = 0, 1, 2, \cdots$, be an increasing sequence of random jump times, such that $\{\tau_{i,j}^{(N)} \mid j = 0, 1, 2, \cdots\}$, $i = 1, 2, \cdots, N$, are independent (independence among particles), $\tau_{i,0}^{(N)} = 0$ and $\{\tau_{i,j+1}^{(N)} - \tau_{i,j}^{(N)} \mid j = 0, 1, 2, \cdots\}$ are i.i.d. with the law of $\tau_i^{(N)} = \tau_{i,1}^{(N)}$ being

$$\mathrm{P}[\, \tau_i^{(N)} \leqq t \,] = 1 - e^{-w_i^{(N)} t},\ t \geqq 0\,. \tag{1}$$

Note that with probability 1, $\tau_{i,j}^{(N)}$, $j = 0, 1, 2, \cdots$, is strictly increasing, and that $\tau_{i,j}^{(N)} \neq \tau_{i',j'}^{(N)}$ for any different pair of suffices $(i, j) \neq (i', j')$.

For each $i = 1, 2, \cdots, N$ we define the time evolution of $X_i^{(N)}$ by,

$$X_i^{(N)}(t) = x_{i,0}^{(N)} + \sharp\{i' \in \{1, 2, \cdots, N\} \mid x_{i',0}^{(N)} > x_{i,0}^{(N)},\ \tau_{i',1}^{(N)} \leqq t\},\quad 0 \leqq t < \tau_{i,1}^{(N)}, \tag{2}$$

where $\sharp A$ denotes the number of elements in the set A, with $\sharp\emptyset = 0$, and for each $j = 1, 2, 3, \cdots$

$$\begin{aligned} &X_i^{(N)}(\tau_{i,j}^{(N)}) = 1, \quad \text{and} \\ &X_i^{(N)}(t) = \sharp\{i' \in \{1, 2, \cdots, N\} \mid \exists j' \in \mathbb{Z}_+;\ \tau_{i,j}^{(N)} < \tau_{i',j'}^{(N)} \leqq t\},\quad \tau_{i,j}^{(N)} < t < \tau_{i,j+1}^{(N)}. \end{aligned} \tag{3}$$

Intuitively speaking, the definition says that particle i jumps at random times $\tau_{i,j}$ to the top of the queue, and that after the jump it is pushed toward the tail every time another particle of larger ranking number jumps to the top. For example, let $N = 4$ and let the initial ranking be $x_{1,0}^{(4)} = 2$, $x_{2,0}^{(4)} = 3$, $x_{3,0}^{(4)} = 1$, $x_{4,0}^{(4)} = 4$. In other words, particles 1–4 are initially aligned as 3124. For a sample ω such that

$$0 < \tau_{1,1}(\omega) < \tau_{2,1}(\omega) < \tau_{4,1}(\omega) < \tau_{1,2}(\omega) < \cdots,$$

the configuration evolves as

$$3124 \ \rightarrow\ 1324 \ \rightarrow\ 2134 \ \rightarrow\ 4213 \ \rightarrow\ 1423 \ \rightarrow\ \cdots,$$

where the changes occur at each jump times $\tau_{i,j}(\omega)$.

The stochastic ranking process may be viewed as a mathematical model of the time evolution of rankings such as that of books on the online bookstores' web (e.g., www.Amazon.co.jp). In this example, N stands for the total number of books, i represents a specific title of a book, $w_i^{(N)}$ is the average rate with which the book i is sold, $x_{i,0}^{(N)}$ is the initial position (ranking) of the book, $\tau_{i,j}^{(N)}$ is the random time at which the book i is sold for the j-th time, and $X_i^{(N)}(t)$ is the ranking of the book i at time t.

In the time interval $(\tau_{i,j}^{(N)}, \tau_{i,j+1}^{(N)})$ the ranking $X_i^{(N)}(t)$ increases by 1 every time one of the books in the tail side of the ranking (i.e., with larger $X_{i'}^{(N)}(t)$) is sold. In other words we have the following.

Proposition 1 *(2) and (3) are equivalent to the following: For each i,* *(i)* $X_i^{(N)}(\tau_{i,j}^{(N)}) = 1,\ j = 1, 2, \cdots,$

(ii) for each $i' \neq i$ *and* $j' = 1, 2, \cdots,$ *if* $X_i^{(N)}(\tau_{i',j'}^{(N)} - 0) < X_{i'}^{(N)}(\tau_{i',j'}^{(N)} - 0)$ *then* $X_i^{(N)}(\tau_{i',j'}^{(N)}) = X_i^{(N)}(\tau_{i',j'}^{(N)} - 0) + 1,$

(iii) otherwise $X_i^{(N)}(t)$ *is constant in* t. ◇

As seen in Proposition 1, each particle jumps at random times to rank 1, and gradually moves to the right (increasing number) without outpacing any other particles on its right. This implies that for each t there is a boundary position $x_C^{(N)}(t) \in \{0, 1, \cdots, N-1\}$ such that all the particles on the left side have experienced a jump, and that none of the particles on the right has jumped by time t:

$$X_i^{(N)}(t) < x_C^{(N)}(t) \Rightarrow \tau_i^{(N)} \leqq t,$$
$$X_i^{(N)}(t) \geqq x_C^{(N)}(t) \Rightarrow \tau_i^{(N)} > t.$$

$x_C^{(N)}(t)$ is a random variable and is explicitly written as:

$$x_C^{(N)}(t) = \sum_{i=1}^{N} \chi_{\tau_i^{(N)} \leqq t}\,. \tag{4}$$

Put

$$y_C^{(N)}(t) = \frac{1}{N} x_C^{(N)}(t) = \frac{1}{N} \sum_{i=1}^{N} \chi_{\tau_i^{(N)} \leqq t}\,. \tag{5}$$

1.2 Motivation.

We are interested in the large N limit of the stochastic ranking process. Noting that (1) implies $\mathrm{E}[\ \chi_{\tau_i^{(N)} \leqq t}\ ] = \mathrm{P}[\ \tau_i^{(N)} \leqq t\ ] = 1 - e^{-w_i^{(N)} t}$, the weak law of large numbers easily leads to the following:

Proposition 2 *Assume that the empirical distribution of jump rates converges to a probability distribution* λ*:*

$$\lambda^{(N)}(dw) = \frac{1}{N} \sum_{i=1}^{N} \delta(w - w_i^{(N)})\, dw \to \lambda(dw),\ \ N \to \infty. \tag{6}$$

Then

$$\lim_{N\to\infty} y_C^{(N)}(t) = y_C(t), \quad \textit{in probability,}$$

where

$$y_C(t) = 1 - \int_{[0,\infty)} e^{-wt} \lambda(dw), \quad t \geqq 0. \tag{7}$$

◇

In the case of the online bookstore Amazon.com, the rankings seem to be defined in a more involved way, but the trajectories of rankings as predicted by Proposition 2 are actually observed at Amazon.co.jp [3, 4]. As may be seen from this example, the stochastic ranking process would be of increasing practical interest and significance in this age of online retails and web 2.0, in analyzing long tail structures [1].

In this paper we go further and prove that in the infinite particle limit $N \to \infty$, the random empirical distribution of the particle system converges to a deterministic space-time dependent distribution. To consider the limit $N \to \infty$, it is natural to use the spacially scaled variables:

$$Y_i^{(N)}(t) = \frac{1}{N} (X_i^{(N)}(t) - 1). \tag{8}$$

$Y_i^{(N)}(t)$ denotes the spacially scaled position of the particle i at time t, taking values in $[0,1) \cap N^{-1}\mathbb{Z}$. In the following, we will use $Y_i^{(N)}(t)$ instead of $X_i^{(N)}(t)$. Correspondingly, we will use

$$y_{i,0}^{(N)} = \frac{1}{N}\,(x_{i,0}^{(N)} - 1) \in [0,1) \cap N^{-1}\mathbb{Z},$$

for the initial configurations instead of $x_{i,0}^{(N)}$.

Main result (informal statement). Under the assumptions on the initial configurations in Section 2.1, the joint random empirical distribution of jump rates (particle types) and positions associated with the stochastic ranking process $\{Y_i^{(N)}\}$ converges as $N \to \infty$ to a distribution (with deterministic time evolution). $\diamond$

The exact mathematical statement of this result is given in Theorem 5, which also contains the explicit form of the limit distribution (see Section 2.2).

In the ranking of books, each time a book is sold its ranking jumps to 1, no matter how unpopular the book may be. At first thought one might guess that such a naive ranking will not be a good index for the popularity of books. But thinking more carefully, one notices that the well sold books (particles with large $w_i^{(N)}$, in our definition) are dominant near the top position, while books near the tail position are rarely sold. Though the rankings of each book are stochastic and exhibit sudden jumps, the *spacial distribution* of the jump rates is more stable, with the large jump rates predominant near the top position. In the bookstore's view, what matters is not a specific book, but the totality of sales. This justifies the interest on the evolution of the empirical distribution of jump rates, as described by Theorem 5.

The limit in Theorem 5 is mathematically non-trivial in that it involves the law of large numbers for dependent variables. Dependence occurs because, for each particle i, the time evolution between the jump times $\tau_i^{(N)}$ is a trajectory of a flow caused by the jumps of other particles in the tail side of the ranking, and the conditioning on tail side induces dependence of stochastic variables.

The idea of considering such a limit theorem is mathematically motivated by the celebrated theory of hydrodynamic limits [5, 6, 7, 8], although the dynamics (relaxation to equilibrium) and hence the proofs in Section 3 apparently have little in common with those of that theory (and are simpler). A difference lies in that the theory of hydrodynamic limits (among other things) evaluates the relaxation to equilibrium (invariant measures) through entropy and large deviation arguments via local equilibrium, while the dynamics of the stochastic ranking process has a special feature that the queue of the particles consists of the 'tail' regime and the 'head' regime, such that the former is the queue of books which has not been sold up to time t, and having no dynamics for relaxation, keeps the remnant of initial data (Section 3.3). In contrast, in the 'head' regime the 'stationarity' is reached from the beginning (Section 3.2). It may also be worthwhile to note that our limit distributions, unlike the hydrodynamic limits which satisfy diffusion equations, satisfy non-local field equations (see remarks to Theorem 5 in Section 2).

In Section 2 we state our main theorem and in Section 3 we give a proof.

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2 Main result.

2.1 Assumptions on initial configuration.

We consider the $N \to \infty$ limit of the empirical distribution on the product space of jump rate and spacial position $\mathbb{R}_+ \times [0,1)$,

$$\mu_t^{(N)}(dw,dy) = \frac{1}{N}\sum_i \delta(w-w_i^{(N)})\delta(y-Y_i^{(N)}(t))\,dw\,dy. \tag{9}$$

We impose that the initial distribution

$$\mu_0^{(N)}(dw,dy) = \frac{1}{N}\sum_i \delta(w-w_i^{(N)})\delta(y-y_{i,0}^{(N)})\,dw\,dy \tag{10}$$

converges weakly as $N \to \infty$ to a probability distribution μ_0 whose second marginal is the Lebesgue measure on $[0,1)$.

For $y \in [0,1)$ let $\mu_{y,0}$ be a regular conditional distribution of μ_0 given y. Then the weak convergence and the continuity of the second marginal imply that for any bounded continuous function $g:\ \mathbb{R}_+ \to \mathbb{R}$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} g(w_i^{(N)})\chi_{y_{i,0}^{(N)}\leqq y} = \int_0^y \left(\int_0^\infty g(w)\mu_{z,0}(dw)\right)\,dz, \tag{11}$$

where (and also in the following) we use a notation χ_A which is 1 if A is true and 0 if A is false. Dini's Theorem implies (for non-negative g, and then with linearity, for any bounded continuous g)

$$\lim_{N\to\infty}\sup_{y\in[0,1)}\left|\frac{1}{N}\sum_{i=1}^{N} g(w_i^{(N)})\chi_{y_{i,0}^{(N)}\leqq y} - \int_0^y \left(\int_0^\infty g(w)\mu_{z,0}(dw)\right)\,dz\right| = 0. \tag{12}$$

Note that $\lambda^{(N)}$ and λ in (6) are the marginal distributions of $\mu_{y,0}^{(N)}$ and $\mu_{y,0}$ of the jump rate;

$$\begin{aligned}\lambda^{(N)}(dw) &= \frac{1}{N}\sum_{i=1}^{N}\delta(w-w_i^{(N)})\,dw = \mu_0^{(N)}(dw,[0,1)),\\ \lambda(dw) &= \int_0^1 \mu_{y,0}(dw)\,dy.\end{aligned} \tag{13}$$

Note also that (12) and Fubini's Theorem imply (6).

We assume that the average of λ is finite,

$$\int w\lambda(dw) < \infty, \tag{14}$$

and

$$\lambda(\{0\}) = 0. \tag{15}$$

This completes the assumptions for our main results.

Remarks. (i) The assumption (14) assures that $\mu_{0,t}$ is well-defined (see (24)). The main results on the existence of the infinite particle limit will hold without (14) for $y > 0$, but we keep this assumption to include $y = 0$.

(ii) We assume (15) to assure that $y_C:\ [0,\infty) \to [0,1)$ defined in (7) is onto (see Proposition 3). The basic results in this paper will hold (with extra complexity in notations and arguments) without (15), but we prefer to keep notations and arguments simple by keeping this assumption. (15) implies in the actual bookstore ranking, that 'all the books sell (almost surely)'. ◇

2.2 Main theorem.

With (15), it is straightforward to show

Proposition 3 *Assume (15). Then* $y_C : [0,\infty) \to [0,1)$ *defined in (7) is a continuous, strictly increasing, bijective function of* t. ◇

Proposition 3 implies the existence of the inverse function $t_0 : [0,1) \to [0,\infty)$, satisfying

$$y_C(t_0(y)) = y, \quad 0 \leqq y < 1, \tag{16}$$

or

$$y = 1 - \int_0^\infty e^{-w t_0(y)} \lambda(dw). \tag{17}$$

Differentiating (7) and (16), we have

$$\frac{d\, y_C}{dt}(t) = \int_0^\infty w e^{-wt} \lambda(dw) = \frac{1}{\dfrac{d\, t_0}{dy}(y_C(t))}. \tag{18}$$

We generalize (7) and define (with slight abuse of notations)

$$y_C(y,t) = 1 - \int_y^1 \int_0^\infty e^{-wt} \mu_{z,0}(dw)\, dz, \quad t \geqq 0,\ 0 \leqq y < 1. \tag{19}$$

In particular, $y_C(t) = y_C(0,t)$. In the infinite particle limit, $y_C(y,t)$ denotes the position of a particle at time t (if it does not jump up to time t) whose initial position is y (Proposition 7).

Proposition 4 $y_C(\cdot,t) : [0,1) \to [y_C(t),1)$ *is a continuous, strictly increasing, bijective function of* y. ◇

Proof. It is straightforward from the definition of $y_C(y,t)$ in (19) to see that $y_C(\cdot,t)$ is continuous and non-decreasing in y. To see that it is strictly increasing, let $0 \leqq z_2 < z_1 < 1$. Then (19) implies $y_C(z_1,t) - y_C(z_2,t) = \int_{z_2}^{z_1} \int_0^\infty e^{-wt} \mu_{z,0}(dw)\, dz$. If this is 0, then $\mu_{z,0}([0,M]) = 0$ for any $M > 0$, for a.e. $z \in [z_2, z_1]$, which contradicts that $\mu_{z,0}$ is a probability measure. □

Proposition 4 implies that the inverse function $\hat{y}(\cdot,t) : [y_C(t),1) \to [0,1)$ exists:

$$1 - y = \int_{\hat{y}(y,t)}^1 \int_0^\infty e^{-wt} \mu_{z,0}(dw)\, dz, \quad t \geqq 0,\ y_C(t) \leqq y < 1. \tag{20}$$

$\hat{y}(y,t)$ denotes the initial position of a particle located at y ($> y_C(t)$) at time t. It holds that

$$\frac{\partial\, \hat{y}}{\partial y}(y,t) = \frac{1}{\displaystyle\int_0^\infty e^{-wt} \mu_{\hat{y}(y,t),0}(dw)}. \tag{21}$$

Now we return to our N-particle process.

Theorem 5 *Consider the stochastic ranking process* $\{Y_i^{(N)}\}$ *defined by (1) and (8). Assume (12) (14) and (15). Then the joint empirical distribution of particle types and positions at time* t

$$\mu_t^{(N)}(dw,dy) = \frac{1}{N} \sum_i \delta(w - w_i^{(N)}) \delta(y - Y_i^{(N)}(t))\, dw\, dy \tag{22}$$

converges as $N \to \infty$ to a distribution $\mu_{y,t}(dw)\,dy$ on $\mathbb{R}_+ \times [0,1)$, that is, for any bounded continuous function $f:\ \mathbb{R}_+ \times [0,1) \to \mathbb{R}$

$$\lim_{N\to\infty} \frac{1}{N} \sum_i f(w_i^{(N)}, Y_i^{(N)}(t)) = \int_0^1 \left(\int_0^\infty f(w,y)\mu_{y,t}(dw) \right) dy, \quad \textit{in probability.} \tag{23}$$

The measure $\mu_{y,t}(dw)$ is given by

$$\mu_{y,t}(dw) = \begin{cases} \dfrac{we^{-wt_0(y)}\lambda(dw)}{\displaystyle\int_0^\infty \tilde{w}e^{-\tilde{w}t_0(y)}\lambda(d\tilde{w})}, & y < y_C(t), \\ \dfrac{e^{-wt}\mu_{\hat{y}(y,t),0}(dw)}{\displaystyle\int_0^\infty e^{-\tilde{w}t}\mu_{\hat{y}(y,t),0}(d\tilde{w})}, & y > y_C(t). \end{cases} \tag{24}$$

◇

Remarks. (i) (22) and (13) imply $\lambda^{(N)}(\cdot) = \mu_t^{(N)}(\cdot,[0,1))$. Moreover, if in (23) we take f without y dependence and use (6), we have as a generalization of (13)

$$\lambda = \int_0^1 \mu_{y,t}\,dy. \tag{25}$$

(ii) Our results state that a random phenomenon approaches a deterministic one as the particle number N is increased. We state the results in terms of convergence in probability, but since the limit quantity is deterministic, this limit is equivalent to convergence in law.

(iii) The explicit forms in (24) differ drastically for $y > y_C(t)$ and $y < y_C(t)$. As we have pointed out in the Introduction, and also as we will see in the proofs in the next section, the dynamics for large y and small y are different.

(iv) By direct calculations, one sees that $\mu_{y,t}(dw)$ satisfies the following equations:

$$\frac{\partial\,\mu_{y,t}(dw)}{\partial t} + \frac{\partial\,(v(y,t)\,\mu_{y,t}(dw))}{\partial y} = -w\mu_{y,t}(dw), \tag{26}$$

where

$$v(y,t) = \int_y^1 \left(\int w\,\mu_{z,t}(dw) \right) dz = \begin{cases} \dfrac{\partial\,y_C}{\partial t}(t_0(y)), & y < y_C(t), \\ \dfrac{\partial\,y_C}{\partial t}(\hat{y}(y,t),t), & y > y_C(t). \end{cases} \tag{27}$$

The non-linear partial differential equation (26) can be seen as the equations of continuity (conservation of mass) for the one-dimensional incompressible mixed fluids, with w standing for the rate of evaporation of specific type of fluid in the mixture [3]. $v(y,t)$ is the velocity of the fluid at position y and time t, and (27), the source of non-locality, means that the flow is driven by evaporation. Intuitively, the equations are natural classical limit of the stochastic processes considered in this paper.

In fact, for the case where λ has a countable support (i.e., in the case of finite or countable types of fluid in the mixture), we can directly prove (without referring to stochastic processes) that (24) is the unique classical solution to the Cauchy problem of partial differential equation (26) with suitable boundary conditions. See [3] for details. ◇

3 Proof of Theorem 5.

It is sufficient to consider the case that $f : \mathbb{R}_+ \times [0,1) \to \mathbb{R}$ in (23) is expressed as $f(w,z) = g(w)\chi_{z\in[0,y]}$, with a bounded continuous function $g : \mathbb{R}_+ \to \mathbb{R}$ and $0 < y < 1$. Thus we prove in this section

$$\lim_{N\to\infty} \frac{1}{N} \sum_i g(w_i^{(N)}) \chi_{Y_i^{(N)}(t) \leqq y} = \int_0^y dz \int_0^\infty g(w) \mu_{z,t}(dw), \quad \text{in probability,} \tag{28}$$

for any bounded continuous function $g : \mathbb{R}_+ \to \mathbb{R}$.

3.1 Case '$y = y_C(t)$'.

Lemma 6 *For each $t > 0$ and each bounded continuous function $g : \mathbb{R}_+ \to \mathbb{R}$, it holds that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_i g(w_i^{(N)}) \chi_{Y_i^{(N)}(t) \leqq y_C^{(N)}(t)} = \int_0^\infty g(w)(1 - e^{-wt})\lambda(dw), \quad \textit{in probability,} \tag{29}$$

and

$$\lim_{N\to\infty} \left(\frac{1}{N} \sum_i g(w_i^{(N)}) \chi_{Y_i^{(N)}(t) \leqq y_C^{(N)}(t)} - \frac{1}{N} \sum_i g(w_i^{(N)}) \chi_{Y_i^{(N)}(t) \leqq y_C(t)} \right) = 0, \quad \textit{in probability.} \tag{30}$$

◇

Proof. The definition of $Y_i^{(N)}(t)$ and (5) imply

$$\chi_{Y_i^{(N)}(t) \leqq y_C^{(N)}(t)} = \chi_{\tau_i^{(N)} \leqq t}. \tag{31}$$

Thus from (1) we have

$$\mathrm{E}[\, g(w_i^{(N)}) \chi_{Y_i^{(N)}(t) \leqq y_C^{(N)}(t)} \,] = g(w_i^{(N)})(1 - e^{-w_i^{(N)} t}).$$

Since g is bounded, in a similar manner as the proof of Proposition 2, we apply the weak law of large numbers to obtain

$$\lim_{N\to\infty} \left(\frac{1}{N} \sum_i g(w_i^{(N)}) \chi_{Y_i^{(N)}(t) \leqq y_C^{(N)}(t)} - \frac{1}{N} \sum_i g(w_i^{(N)})(1 - e^{-w_i^{(N)} t}) \right) = 0, \quad \text{in probability.}$$

By (6) we have (29). Next, Proposition 2 implies that, for any $\epsilon > 0$, with large enough N, it holds that

$$\mathrm{P}[\, |y_C(t) - y_C^{(N)}(t)| \leqq \epsilon \,] > 1 - \epsilon.$$

Note that

$$\sum_i |\chi_{Y_i^{(N)}(t) \leqq y_C^{(N)}(t)} - \chi_{Y_i^{(N)}(t) \leqq y_C(t)}| = \sum_{k=1}^N |\chi_{k \leqq N y_C^{(N)}(t)} - \chi_{k \leqq N y_C(t)}| \leqq N |y_C^{(N)}(t) - y_C(t)| + 1,$$

so that

$$\begin{aligned} &\left| \frac{1}{N} \sum_i g(w_i^{(N)}) \chi_{Y_i^{(N)}(t) \leqq y_C^{(N)}(t)} - \frac{1}{N} \sum_i g(w_i^{(N)}) \chi_{Y_i^{(N)}(t) \leqq y_C(t)} \right| \\ &\leqq \frac{1}{N} \sup_w |g(w)| \left(|y_C^{(N)}(t) - y_C(t)| + \frac{1}{N} \right). \end{aligned}$$

Hence for large N

$$\begin{aligned} &\mathrm{P}\left[\left| \frac{1}{N} \sum_i g(w_i^{(N)}) \chi_{Y_i^{(N)}(t) \leqq y_C^{(N)}(t)} - \frac{1}{N} \sum_i g(w_i^{(N)}) \chi_{Y_i^{(N)}(t) \leqq y_C(t)} \right| \leqq \sup_w |g(w)| (\epsilon + \frac{1}{N}) \,\right] \\ &\geqq \mathrm{P}[\, |y_C(t) - y_C^{(N)}(t)| \leqq \epsilon \,] > 1 - \epsilon, \end{aligned}$$

which implies (30). □

3.2 Case $y < y_C(t)$.

First note that, to prove (28) for $y < y_C(t)$, it is sufficient to prove that for each bounded continuous function $g:\ \mathbb{R}_+ \to \mathbb{R}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_i g(w_i^{(N)}) \chi_{Y_i^{(N)}(t) \leqq y} = \int_0^\infty g(w)(1 - e^{-wt_0(y)})\lambda(dw), \quad \text{in probability,} \tag{32}$$

where $t_0(y)$ is as in (16). To see that (32) implies (28), differentiate the right-hand side of (32) with respect to y, use (18) and (16), and integrate from 0 to y, keeping in mind $t_0(0) = 0$, and finally rewrite using $\mu_{y,t}(dw)$ in (24), to obtain

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N} \sum_i g(w_i^{(N)}) \chi_{Y_i^{(N)}(t) \leqq y} &= \int_0^y \frac{\displaystyle\int_0^\infty g(w) w e^{-wt_0(z)} \lambda(dw)}{\displaystyle\int_0^\infty w e^{-wt_0(z)} \lambda(dw)}\, dz \\ = \int_0^y \int g(w) \mu_{z,t}(dw)\, dz, &\quad \text{in probability,}\end{aligned}$$

which gives (28).

To prove (32), fix $y < y_C(t)$ and let $t_0 = t_0(y) < t$.

Denote by $\{\tilde{Y}_i^{(N)}(s), \tilde{\tau}_{i,j}^{(N)}\}$ the scaled stochastic ranking process with the time origin shifted by the amount $t - t_0 > 0$. Namely, let $\tilde{Y}_i^{(N)}(s) = Y_i^{(N)}(s + t - t_0)$. In particular, we have

$$\frac{1}{N} \sum_i g(w_i^{(N)}) \chi_{Y_i^{(N)}(t) \leqq y} = \frac{1}{N} \sum_i g(w_i^{(N)}) \chi_{\tilde{Y}_i^{(N)}(t_0) \leqq y}\,. \tag{33}$$

For $j = 0, 1, 2, \cdots$, define $\tilde{\tau}_{i.j}^{(N)}$ by $\tilde{\tau}_{i,0}^{(N)} = 0$ and

$$\tilde{\tau}_{i.j}^{(N)} = \tau_{i,\tilde{j}(i,t-t_0)+j-1}^{(N)} - (t - t_0)\,, \tag{34}$$

where

$$\tilde{j}(i, t - t_0) = \inf\{j \mid \tau_{i,j}^{(N)} > t - t_0\}. \tag{35}$$

Put, in analogy to (5),

$$\tilde{y}_C^{(N)}(s) = \frac{1}{N} \sum_{i=1}^N \chi_{\tilde{\tau}_i^{(N)} \leqq s}\,.$$

Note that $\{\tau_{i,j+1}^{(N)} - \tau_{i,j}^{(N)} \mid j = 0, 1, 2, \cdots\}$ are independent and have exponential distributions. The loss of memory property of exponential distributions then implies that $\{\tilde{\tau}_{i,j}^{(N)}\}$ have the same distributions as $\{\tau_{i,j}^{(N)}\}$, and $\{\tilde{Y}_i^{(N)}(s)\}$ is a scaled stochastic ranking process with jump times $\{\tilde{\tau}_{i,j}^{(N)}\}$ and initial configuration $\{Y_i^{(N)}(t - t_0)\}$.

Since $y = y_C(t_0)$, (30) for the time shifted ranking process implies

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N} \sum_i g(w_i^{(N)}) \chi_{\tilde{Y}_i^{(N)}(t_0) \leqq y} &= \lim_{N\to\infty} \frac{1}{N} \sum_i g(w_i^{(N)}) \chi_{\tilde{Y}_i^{(N)}(t_0) \leqq y_C(t_0)} \\ = \lim_{N\to\infty} \frac{1}{N} \sum_i g(w_i^{(N)}) \chi_{\tilde{Y}_i^{(N)}(t_0) \leqq \tilde{y}_C^{(N)}(t_0)}, &\quad \text{in probability.}\end{aligned} \tag{36}$$

Using (31) for the original process and the time shifted process, and recalling that $\{\tau_i^{(N)}\}$ and $\{\tilde{\tau}_i^{(N)}\}$ have the same distribution, and then using (29), we arrive at

$$\begin{aligned} &\lim_{N\to\infty}\frac{1}{N}\sum_i g(w_i^{(N)})\chi_{\tilde{Y}_i^{(N)}(t_0)\leqq\tilde{y}_C^{(N)}(t_0)} = \lim_{N\to\infty}\frac{1}{N}\sum_i g(w_i^{(N)})\chi_{\tilde{\tau}_i^{(N)}\leqq t_0} \\ &= \lim_{N\to\infty}\frac{1}{N}\sum_i g(w_i^{(N)})\chi_{\tau_i^{(N)}\leqq t_0} = \lim_{N\to\infty}\frac{1}{N}\sum_i g(w_i^{(N)})\chi_{Y_i^{(N)}(t_0)\leqq y_C^{(N)}(t_0)} \\ &= \int_0^\infty g(w)(1-e^{-wt_0})\lambda(dw), \quad \text{in probability.} \end{aligned} \tag{37}$$

(33) (36) and (37) together imply (32).

3.3 Case $y > y_C(t)$.

First we make some preparations for the proof. We generalize $y_C^{(N)}(t)$ in (5) and define (with slight abuse of notations) for $0 \leqq y < 1$

$$y_C^{(N)}(y,t) = y + \frac{1}{N}\sum_{i=1}^N \chi_{\tau_i^{(N)}\leqq t}\chi_{y_{i,0}^{(N)}\geqq y}. \tag{38}$$

Proposition 7 *For $0 \leqq y < 1$ and $t \geqq 0$,*

$$\lim_{N\to\infty} y_C^{(N)}(y,t) = y_C(y,t), \quad \textit{in probability.} \tag{39}$$

Namely, the (random) position $y_C^{(N)}(y,t)$ of a particle at time t whose initial position is y converges in probability to a deterministic trajectory $y_C(y,t)$ defined by (19) in the infinite particle limit. $\diamond$

Remark. The proof below shows that the convergence in (39) holds uniformly in y. $\diamond$

Proof. (38) and (1) and the independence of $\{\tau_i^{(N)}\}$ imply

$$\begin{aligned} &\mathrm{E}[\ \left(y_C(y,t) - y_C^{(N)}(y,t)\right)^2\ ] \\ &= y_C(y,t)^2 - 2y_C(y,t)\left(y + \frac{1}{N}\sum_{i=1}^N (1-e^{-w_i^{(N)}t})\chi_{y_{i,0}^{(N)}\geqq y}\right) \\ &+ y^2 + 2y\frac{1}{N}\sum_{i=1}^N (1-e^{-w_i^{(N)}t})\chi_{y_{i,0}^{(N)}\geqq y} \\ &+ \frac{1}{N^2}\sum_{i=1}^N\sum_{j=1}^N (1-e^{-w_i^{(N)}t})(1-e^{-w_j^{(N)}t})\chi_{y_{i,0}^{(N)}\geqq y}\chi_{y_{j,0}^{(N)}\geqq y} \\ &+ \frac{1}{N^2}\sum_{i=1}^N \left(1-e^{-w_i^{(N)}t} - (1-e^{-w_i^{(N)}t})^2\right)\chi_{y_{i,0}^{(N)}\geqq y} \\ &= \left(y + \frac{1}{N}\sum_{i=1}^N (1-e^{-w_i^{(N)}t})\chi_{y_{i,0}^{(N)}\geqq y} - y_C(y,t)\right)^2 \\ &+ \frac{1}{N^2}\sum_{i=1}^N \left(1-e^{-w_i^{(N)}t} - (1-e^{-w_i^{(N)}t})^2\right)\chi_{y_{i,0}^{(N)}\geqq y}. \end{aligned}$$

The second term in the right-hand side of the equation above vanishes in the $N \to \infty$ limit because of the factor N^2 in the denominator. Concerning the first term, as in the proof of Proposition 2 and Lemma 6, (12) implies that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} (1 - e^{-w_i^{(N)} t}) \chi_{y_{i,0}^{(N)} \geqq y} = \int_y^1 \int_0^\infty (1 - e^{-wt}) \mu_{z,0}(dw)\, dz$$
$$= 1 - y - \int_y^1 \int_0^\infty e^{-wt} \mu_{z,0}(dw)\, dz,$$

uniformly in y. This combined with (19) implies that the first term also vanishes. Thus we have

$$\lim_{N\to\infty} \mathrm{E}[\ \left(y_C(y,t) - y_C^{(N)}(y,t) \right)^2\ ] = 0.$$

With Chebyshev's inequality follows (39). □

As an equivalent statement to (28), we will prove

$$\lim_{N\to\infty} \frac{1}{N} \sum_i g(w_i^{(N)}) \chi_{Y_i^{(N)}(t) > y} = \int_y^1 dz \int_0^\infty g(w) \mu_{z,t}(dw), \quad \text{in probability.} \tag{40}$$

Let $\Omega_1^{(N)} = \{ y > y_C^{(N)}(t) \}$. Then Proposition 2 implies that if $y > y_C(t)$,

$$\lim_{N\to\infty} \mathrm{P}[\ \Omega_1^{(N)}\ ] = 1. \tag{41}$$

For $t > 0$ and $y > y_C^{(N)}(t)$, let

$$\hat{y}^{(N)}(y,t) = \inf\{ y_{i,0}^{(N)} \mid i = 1, \cdots, N,\ Y_i^{(N)}(t) > y \}. \tag{42}$$

Note that

$$|y - y_C^{(N)}(\hat{y}^{(N)}(y,t), t)| \leqq \frac{2}{N}\,. \tag{43}$$

This follows because

$$Y_i^{(N)}(t) = y_C^{(N)}(y_{i,0}^{(N)}, t), \quad \text{if } Y_i^{(N)}(t) > y_C^{(N)}(t), \tag{44}$$

hence with $y > y_C^{(N)}(t)$

$$y - \frac{1}{N} \leqq y_C^{(N)}(\hat{y}^{(N)}(y,t), t) \leqq Y_i^{(N)}(t) + \frac{1}{N}$$

for all i with $Y_i^{(N)}(t) > y$.

Until a particle jumps to the top of the queue, changes of its position are caused only by the jumps of other particles that sit on its right (Proposition 1), hence

$$\sum_i g(w_i^{(N)}) \chi_{Y_i^{(N)}(t) > y} = \sum_i g(w_i^{(N)}) \chi_{y_{i,0}^{(N)} \geqq \hat{y}^{(N)}(y,t)} \chi_{\tau_i^{(N)} > t}, \quad \text{on } \Omega_1^{(N)}. \tag{45}$$

Note that $\hat{y}^{(N)}(y,t)$ depends on τ_i's. This means that the summands on the right-hand side are not independent random variables, and that we can not apply the law of large numbers as it is. In contrast, since $\hat{y}(y,t)$ is deterministic, the law of large numbers yields, as in the proofs of Proposition 2 and Lemma 6,

$$\lim_{N\to\infty} \frac{1}{N} \sum_i g(w_i^{(N)}) \chi_{y_{i,0}^{(N)} \geqq \hat{y}(y,t)} \chi_{\tau_i^{(N)} > t} = \int_{\hat{y}(y,t)}^1 \int_0^\infty g(w) e^{-wt} \mu_{z,0}(dw)\, dz, \quad \text{in probability.} \tag{46}$$

The right-hand side coincides with that of (40) through a change of variables $\hat{y}(z,t) \to z$ (note (21)). Combining (40), (41), (45) and (46), we see that it is sufficient to prove that

$$\lim_{N\to\infty} P[\; y > y_C^{(N)}(t),\quad \frac{1}{N}\sum_i g(w_i^{(N)}) \left| \chi_{y_{i,0}^{(N)} \geqq \hat{y}(y,t)} - \chi_{y_{i,0}^{(N)} \geqq \hat{y}^{(N)}(y,t)} \right| \chi_{\tau_i^{(N)} > t} \geqq \epsilon \;] = 0 \tag{47}$$

holds for any $\epsilon \in (0,1)$.

Note that if $y > y_C^{(N)}(t)$ and $|\hat{y}^{(N)}(y,t) - \hat{y}(y,t)| \leqq \epsilon$, then

$$\frac{1}{N}\sum_i g(w_i^{(N)}) \left| \chi_{y_{i,0}^{(N)} \geqq \hat{y}(y,t)} - \chi_{y_{i,0}^{(N)} \geqq \hat{y}^{(N)}(y,t)} \right| \chi_{\tau_i > t} \leqq \frac{M}{N}\sum_i \left| \chi_{y_{i,0}^{(N)} \geqq \hat{y}(y,t)} - \chi_{y_{i,0}^{(N)} \geqq \hat{y}^{(N)}(y,t)} \right|$$
$$\leqq M\,(\epsilon + \frac{1}{N}),$$

where $M > 0$ is a constant satisfying $|g(w)| \leqq M$ for all $w \geqq 0$. Thus to prove (47), it is sufficient to show

$$\lim_{N\to\infty} \mathrm{P}[\; y > y_C^{(N)}(t),\quad |\hat{y}^{(N)}(y,t) - \hat{y}(y,t)| > \epsilon \;] = 0,$$

for an arbitrary $\epsilon > 0$.

Let us assume otherwise, that is, with (41) in mind, assume that there is $\epsilon_1 > 0$, $\rho > 0$ and an increasing sequence of positive integers $\{N_i\}$ such that

$$\mathrm{P}[\; y > y_C^{(N_i)}(t),\quad \hat{y}^{(N_i)}(y,t) - \hat{y}(y,t) > \epsilon_1 \;] > \rho \tag{48}$$

or

$$\mathrm{P}[\; y > y_C^{(N_i)}(t),\quad \hat{y}^{(N_i)}(y,t) - \hat{y}(y,t) < -\epsilon_1 \;] > \rho$$

holds. We consider the case (48), since the second case is dealt with similarly. Let $y_1 = \hat{y}(y,t)$. With Proposition 4, we have

$$y = y_C(y_1,t) < y_C(y_1+\epsilon_1,t).$$

Let $\epsilon_2 = y_C(y_1+\epsilon_1,t) - y > 0$. Proposition 7 implies

$$\mathrm{P}[\; |y_C^{(N_i)}(y_1+\epsilon_1,t) - y_C(y_1+\epsilon_1,t)| \geqq \frac{\epsilon_2}{4} \;] \leqq \frac{\rho}{2} \tag{49}$$

holds for sufficiently large N_i. Combining (48), $y_1 = \hat{y}(y,t)$ and (49), we have

$$\mathrm{P}[\; y > y_C^{(N_i)}(t),\quad y_1+\epsilon_1 < \hat{y}^{(N_i)}(y,t),\quad |y_C^{(N_i)}(y_1+\epsilon_1,t) - y_C(y_1+\epsilon_1,t)| < \frac{\epsilon_2}{4} \;] > \frac{\rho}{2} > 0$$

for sufficiently large N_i. Note that (38) implies that if $y \leqq y'$ then $y_C^{(N)}(y,t) \leqq y_C^{(N)}(y',t) + \dfrac{1}{N}$. This combined with (43) implies that if $y > y_C^{(N_i)}(t)$ and $y_1+\epsilon_1 < \hat{y}^{(N_i)}(y,t)$, then

$$y_C^{(N_i)}(y_1+\epsilon_1,t) < y + \frac{\epsilon_2}{4} + \frac{3}{N_i}, \tag{50}$$

for sufficiently large N_i. On the other hand, if $|y_C^{(N_i)}(y_1+\epsilon_1,t) - y_C(y_1+\epsilon_1,t)| < \dfrac{\epsilon_2}{4}$, then

$$y + \epsilon_2 = y_C(y_1+\epsilon_1,t) < y_C^{(N_i)}(y_1+\epsilon_1,t) + \frac{\epsilon_2}{4}. \tag{51}$$

But (50) and (51) put together imply

$$y + \epsilon_2 < y + \frac{\epsilon_2}{2} + \frac{3}{N_i},$$

which is a contradiction for large N_i. Thus the assumption (48) is false, which completes the proof of Theorem 5.

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