Loop-erased random walk on the Sierpinski gasket

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ABSTRACT

In this paper the loop-erased random walk on the finite pre-Sierpiński gasket is studied. It is proved that the scaling limit exists and is a continuous process. It is also shown that the path of the limiting process is almost surely self-avoiding, while having Hausdorff dimension strictly greater than 1. The loop-erasing procedure proposed in this paper is formulated by erasing loops, in a sense, in descending order of size. It enables us to obtain exact recursion relations, making direct use of 'self-similarity' of a fractal structure, instead of the relation to the uniform spanning tree. This procedure is proved to be equivalent to the standard procedure of chronological loop-erasure.

Key words: loop-erased random walk ; scaling limit ; displacement exponent ; fractal dimension ; Sierpinski gasket ; fractal

MSC2010 Subject Classifications: 60F99, 60G17

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1 Introduction

The loop-erased random walk (LERW) was first introduced by Lawler on Euclidean lattices $\mathbb{Z}^d$ [14]. It is a process obtained by chronologically erasing loops from a simple random walk. In the following we shall refer to this process as the 'standard' LERW. In this paper, we consider the LERW on the pre-Sierpiński gasket, which lacks translational invariance but has 'self-similarity'.

A natural question to ask will be : How far an n-step LERW can go in average? To be more precise, if $X(n)$ denotes the location of a LERW starting at the origin after n steps, does the mean square displacement show a power behavior? That is,

$$E[|X(n)|^2] \sim n^{2\nu},$$

where $|X(n)|$ denotes the distance from the origin and ν is a positive constant. If true, what is the value of ν? ν is called the **displacement exponent** and $1/\nu$ is referred to as the **fractal**

dimension of LERW. On the pre-Sierpiński gasket, Dhar and Dhar obtained the exact value $\nu = \log 2/\log\{\frac{1}{15}(20+\sqrt{205})\}$ [5], and Shinoda (unpublished) gave it a rigorous proof by showing

$$K_1 n^{2\nu} \leqq E[|X(n)|^2] \leqq K_2 n^{2\nu},$$

where K_1 and K_2 are positive constants. They make use of the relation between LERW and the uniform spanning tree.

Another important question is: Does the LERW have a scaling limit? A scaling limit of random walk on a lattice is a limit as the lattice spacing tends to zero (with appropriate time-scaling). If it exists, what are the properties of the limit process? On $\mathbb{Z}^d$, the scaling limits of LERWs have been known. For $d \geqq 4$, Lawler proved that LERW converges to Brownian motion [15], [16]. For $d = 3$, Kozma proved the existence of the scaling limit [13]. Lawler, Schramm and Werner proved that LERW on $\mathbb{Z}^2$ has a conformal invariant scaling limit, using Schramm Loewner Evolution (SLE) [17], [21]. We remark that the exact fractal dimension $1/\nu = 5/4$ for the LERW on $\mathbb{Z}^2$ was obtained by Majumdar [18] before these SLE results.

In this paper, first we propose a different method of loop-erasure, that is, not by erasing loops in chronological order, but by erasing, in a sense, in descending order of size, which we call an 'erasing-larger-scale-loops-first' rule. This procedure makes it easier to obtain recursion relations, making use of 'self-similarity' of a fractal structure, without using the uniform spanning tree. We prove that the LERW defined here is equivalent to the standard LERW. Generally, if we erase loops in a different order, we get a different path. In fact, it is possible to erase loops in such a way that the resulting walk has a different distribution from the standard LERW. Thus, the equivalence is not trivial.

We shall prove that the scaling limit exists in the sense that a LERW path converges uniformly to a continuous path with probability one. We regard the limit path as a random fractal and show that with probability one it has Hausdorff dimension $1/\nu = \log\{\frac{1}{15}(20+\sqrt{205})\}/\log 2 = 1.1939\ldots$.

We also show that the path of the limiting process is almost surely self-avoiding. Thus, the path of the limit process has infinitely fine creases, while having no self-intersection. Since we are considering a limit, it is not self-evident that the resulting process is also self-avoiding.

There have been studies of the self-avoiding walk (SAW) on the pre-Sierpiński gasket ([3], [12], [20], [7], [8], [9]). The LERW is also self-avoiding in the sense that paths have no self-intersection, but it belongs to a different universality class from SAW, which has fractal dimension $1/\nu_{SAW} = \log 3/\log\{\frac{1}{2}(7-\sqrt{5})\} = 1.2657\ldots$.

In Section 2, we describe the set-up of our model and the loop-erasing procedure. In Section 3, we show the equivalence of LERWs obtained by two different methods of loop-erasing. Section 4 is devoted to the examination of the scaling limit.

2 Paths on the pre-Sierpiński gaskets

2.1 The pre-Sierpiński gaskets.

We consider the pre-Sierpiński gasket, a lattice version of the Sierpiński gasket, which is a fractal with Hausdorff dimension $\log 3/\log 2$. (For fractals, see [6].) Let us recall the definition of the pre-Sierpiński gasket: by denoting $O = (0,0)$, $a_0 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $b_0 = (1,0)$, and for each $N \in \mathbb{N}$, $a_N = 2^N a_0$, $b_N = 2^N b_0$, then define F'_0 be the graph that consists of three vertices and three edges of $\triangle O a_0 b_0$ and define the recursive sequence of graphs $\{F'_N\}_{N=0}^{\infty}$ by

$$F'_{N+1} = F'_N \cup (F'_N + a_N) \cup (F'_N + b_N), \quad N \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\},$$

where $A + a = \{x + a \ : \ x \in A\}$ and $kA = \{kx \ : \ x \in A\}$. F'_0, F'_1 and F'_2 are shown in Fig. 1.

Finally, we let F''_N be the union of F'_N and its reflection with respect to the y-axis, and denote $F_0 = \bigcup_{N=1}^{\infty} F''_N$; the graph F_0 is called the (infinite) **pre-Sierpiński gasket**. F_0 is shown in Fig. 2.

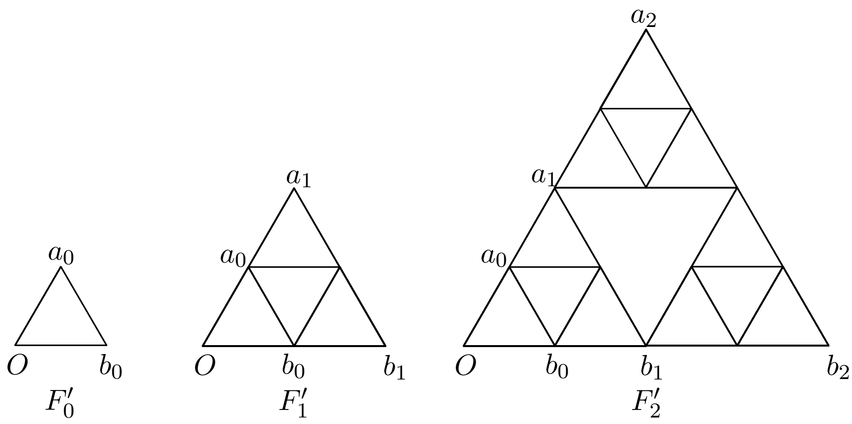

Fig 1: F_0', F_1' and F_2'.

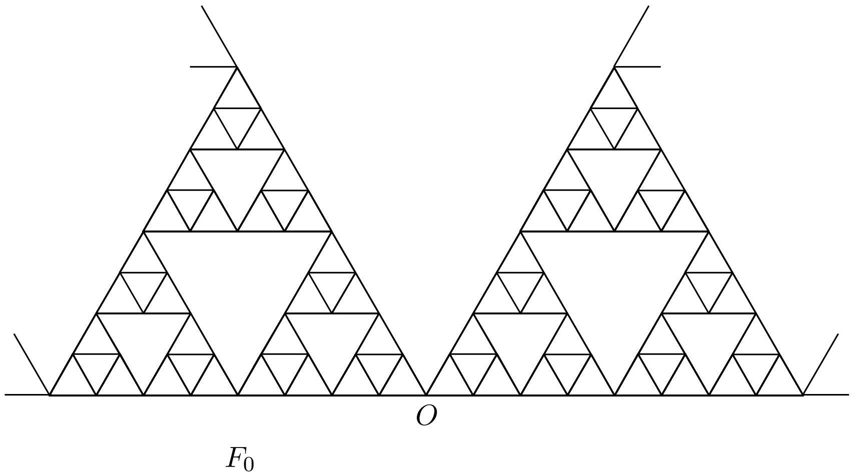

Fig 2: The pre-Sierpiński gasket F_0

Furthermore, by letting G_0 and E_0 denote the set of vertices and the set of edges of F_0, respectively, we see that, for each $N \in \mathbb{Z}_+$, $F_N = 2^N F_0$ can be regarded as a coarse graph with vertices $G_N = \{2^N x \ : \ x \in G_0\}$ and edges $E_N = \{2^N \overline{xy} \ : \ \overline{xy} \in E_0\}$. Given $x \in G_N$, let $\mathcal{N}_N(x)$ be the four nearest neighbors of x on F_N, that is, $\mathcal{N}_N(x) = \{y \in G_N : \ \overline{xy} \in E_N\}$.

2.2 Paths on the pre-Sierpiński gaskets.

Let us denote the set of finite paths on F_0 by

$$W = \{ \ w = (w(0), w(1), \cdots, w(n)) : \ w(0) \in G_0, \ w(i) \in \mathcal{N}_0(w(i-1)),$$

$$1 \leqq i \leqq n, \ n \in \mathbb{N} \ \},$$

and the set of finite paths on F_0 starting at O by

$$W^* = \{ \ w \in W \ : \ w(0) = O \ \}.$$

This gives the natural definition for the length ℓ of a path $w = (w(0), w(1), \ \cdots, w(n)) \in W$; namely, $\ell(w) = n$.

For a path $w \in W$ and $A \subset G_0$, we define the hitting time of A by

$$T_A(w) = \inf\{j \geqq 0 : \ w(j) \in A\},$$

where we set $\inf \emptyset = \infty$. By taking $w \in W$ and $M \in \mathbb{Z}_+$, we shall define the recursive sequence $\{T_i^M(w)\}_{i=0}^m$ of **hitting times of** G_M as follows: Let $T_0^M(w) = T_{G_M}$, and for $i \geqq 1$, let

$$T_i^M(w) = \inf\{j > T_{i-1}^M(w) : \ w(j) \in G_M \setminus \{w(T_{i-1}^M(w))\}\};$$

here we take m to be the smallest integer such that $T_{m+1}^M(w) = \infty$. Then $T_i^M(w)$ can be interpreted as being the time taken for the path w to hit vertices in G_M for the $(i+1)$-th time, under the condition that if w hits the same vertex in G_M more than once in a row, we count it only once.

Now we consider two sequences of subsets of W^* as follows: for each $N \in \mathbb{Z}_+$, let the set of paths from O to a_N, which do not hit any other vertices in G_N on the way, be

$$W_N = \{w = (w(0), w(1), \cdots, w(n)) \in W^* : \ w(n) = a_N, \ n = T_1^N(w)\},$$

and let the set of paths from from O to a_N that hit b_N 'once' on the way (subject to the counting rule explained above) be

$$V_N = \{w = (w(0), w(1), \cdots, w(n)) \in W^* : \ w(n) = a_N,$$

$$w(T_1^N(w)) = b_N, \ n = T_2^N(w)\}.$$

Then for a path $w \in W$ and $M \in \mathbb{Z}_+$, we define the **coarse-graining map** Q_M by

$$(Q_M w)(i) = w(T_i^M(w)), \quad \text{for } i = 0, 1, 2, \ldots, m,$$

where m is the smallest integer such that $T_{m+1}^M(w) = \infty$. Thus,

$$Q_M w = [w(T_0^M(w)), w(T_1^M(w)), \ldots, w(T_m^M(w))]$$

is a path on a coarser graph F_M. For $w \in W_N \cup V_N$ and $M \leqq N$, the end point of the coarse-grained path is $w(T_m^M(w)) = a_N$, and if we write $(2^{-M}Q_M w)(i) = 2^{-M}w(T_i^M(w))$, then $2^{-M}Q_M w$ is a path in $W_{N-M} \cup V_{N-M}$ and $\ell(2^{-M}Q_M w) = m$. Notice that if $M \leqq N$, then $Q_N \circ Q_M = Q_N$. Throughout the following, we write simply $w(T_i^M)$ instead of $w(T_i^M(w))$.

2.3 Loop-erased paths.

Let Γ be the set of self-avoiding paths starting at O:

$$\Gamma = \{ \ (w(0), w(1), \cdots, w(n)) \in W^* : \ w(i) \neq w(j), \ 0 \leqq i < j \leqq n, \ n \in \mathbb{N} \ \},$$

and let us denote the following two subsets of Γ :

$$\hat{W}_N = W_N \cap \Gamma, \quad \hat{V}_N = V_N \cap \Gamma.$$

For $(w(0), w(1), \cdots, w(n)) \in W^*$, if there are i and j, $0 \leqq i < j \leqq n$ such that $w(i) = w(j)$ and $w(k) \neq w(i)$ for any $i < k < j$, we call the path segment $[w(i), w(i+1), \ldots, w(j)]$ a loop.

We shall now describe a loop-erasing procedure for paths in $W_1 \cup V_1$:

(i) Erase all the loops formed at O;

(ii) Progress one step forward along the path, and erase all the loops at the new position;

(iii) Iterate this process, taking another step forward along the path and erasing the loops there, until reaching a_1 (the endpoint of all paths in W_1 and V_1).

To be precise, for $w \in W_1 \cup V_1$, define the recursive sequence $\{s_i\}_{i=0}^n$

$$s_0 = \sup\{j : w(j) = O\},$$

$$s_i = \sup\{j : w(j) = w(s_{i-1}+1)\}.$$

If $s_i > s_{i-1}+1$, then $[w(s_{i-1}+1), w(s_{i-1}+2), \ldots, w(s_i-1), w(s_i)]$ forms a loop, starting and ending at $w(s_{i-1}+1) = w(s_i)$. We erase it by removing all of the points $w(s_{i-1}+1), w(s_{i-1}+2), \ldots, w(s_i-2), w(s_i-1)$. If $w(s_n) = a_1$, then we have obtained a loop-erased path,

$$Lw = [w(s_0), w(s_1), \ldots, w(s_n)] \in \hat{W}_1 \cup \hat{V}_1.$$

Note that $w \in W_1$ implies $Lw \in \hat{W}_1$, but that $w \in V_1$ can result in $Lw \in \hat{W}_1$, with b_1 being erased together with a loop. So far, our loop-erasing procedure is the same as that defined for paths on $\mathbb{Z}^d$ in [15].

We shall generalize the above procedure to a loop-erasing procedure for a path w in $W_N \cup V_N$ that yields a self-avoiding path in $\hat{W}_N \cup \hat{V}_N$. The idea is to first erase loops of 'largest scale', and then go down to 'smaller scales' step by step. For this purpose, we need the notion of 'skeletons'.

Let $\mathcal{T}_M$ be the set of all upward (closed and filled) triangles which are translations of $\triangle O a_M b_M$ and whose vertices are in G_M; an element of $\mathcal{T}_M$ is called a 2^M**-triangle**. For $w \in W$ and $M \geqq 0$, we shall define a sequence $(\Delta_1, \ldots, \Delta_k)$ of the 2^M-triangles w 'passes through' and a sequence $\{T_i^{ex,M}(w)\}_{i=1}^k$ of the exit times from them as a subsequence of $\{T_i^M(w)\}_{i=1}^m$, as follows: We start by defining $T_0^{ex,M}(w) = T_0^M(w)$. (Thus If $w \in W^*$, then $T_0^{ex,M}(w) = 0$.) There is a unique element of $\mathcal{T}_M$ that contains $w(T_0^M)$ and $w(T_1^M)$, which we denote by Δ_1. For $i \geqq 1$, define

$$j(i) = \min\{j \geqq 0 \ : \ j < m,\ T_j^M(w) > T_{i-1}^{ex,M}(w),\ w(T_{j+1}^M(w)) \notin \Delta_i\},$$

if the minimum exists, otherwise $j(i) = m$. Then define $T_i^{ex,M}(w) = T_{j(i)}^M(w)$, and let Δ_{i+1} be the unique 2^M-triangle that contains both $w(T_i^{ex,M})$ and $w(T_{j(i)+1}^M)$. By definition, we see that $\Delta_i \cap \Delta_{i+1}$ is a one-point set $\{w(T_i^{ex,M})\}$, for $i = 1, \ldots, k-1$. We denote the sequence of these triangles by $\sigma_M(w) = (\Delta_1, \ldots, \Delta_k)$, and call it the 2^M**-skeleton** of w. We call the sequence $\{T_i^{ex,M}(w)\}_{i=0,1,\ldots,k}$ **exit times** from the triangles in the skeleton. For each i, there is an $n = n(i)$ such that $T_{i-1}^{ex,M}(w) = T_n^M(w)$. We say $\Delta_i \in \sigma_M(w)$ is an element of **Type 1** if $T_i^{ex,M}(w) = T_{n+1}^M$, and an element of **Type 2** if $T_i^{ex,M}(w) = T_{n+2}^M$. If $w \in \hat{W}_N \cup \hat{V}_N$ for some N, then $\Delta_1, \ldots, \Delta_k$ are mutually distinct, and each of them is either of Type 1 or of Type 2.

Assume $w \in W_N \cup V_N$ for some N and $M \leqq N$. For each Δ in $\sigma_M(w)$, the **path segment of** w **in** Δ is defined by

$$[w(n),\ T_{i-1}^{ex,M}(w) \leqq n \leqq T_i^{ex,M}],$$

and it is denoted by $w|_\Delta$. Note that the definition of T_i^M's allows a path segment $w|_\Delta$ to leak into two neighboring 2^M-triangles.

It should be noted that the subgraph contained in Δ and its neighboring triangles has the same structure as $\triangle O a_M b_M$ and its neighbors, which implies that $w|_\Delta$ can be naturally identified with some path in $\triangle O a_M b_M$ and its neighbors starting at O, by translation, rotation and reflection. For convenience we shall denote this identification by η, and write:

$$\eta(w|_\Delta) = v \in W_M \cup V_M, \tag{2.1}$$

where the entrance to Δ is mapped to O and the exit to a_M.

To introduce the loop-erasing operation for paths in $W_N \cup V_N$, let us take a loop $[w(i), w(i+1), \ldots, w(i+i_0)]$ that is contained in $w \in W_N \cup V_N$, and define its diameter by $d = \max\{i < j \leqq i+i_0 : |w(j) - w(i)|\}$. The loop $[w(i), w(i+1), \ldots, w(i+i_0)]$ is said to be a 2^M**-scale loop**, whenever there exists an $M \in \mathbb{Z}_+$ such that

$$\max\{N' : w(i) = w(i+i_0) \in G_{N'}\} = M \text{ and } d \geqq 2^M.$$

Notice that a path in $W_N \cup V_N$ has no 2^N-scale loops by definition. Then the above definition implies that w has a 2^{N-1}-scale loop if and only if the coarse-grained path $Q_{N-1}w$ has a loop. The operation of erasing largest-scale loops can be reduced to erasing loops from a path in $W_1 \cup V_1$, which we shall show below by induction.

Let $w \in W_N \cup V_N$ (Fig. 3(a)). we define the operation of 'erasing the largest-scale loops' as follows:

1) Coarse-grain w to obtain

$$w' = Q_{N-1}w = [w(T_0^{N-1}), w(T_1^{N-1}), \ldots, w(T_k^{N-1})],$$

where $w(T_k^{N-1}) = a_N$ (Fig. 3(b)). We note that $2^{-(N-1)}w' \in W_1 \cup V_1$.

2) Similarly to the procedure for $W_1 \cup V_1$, erase loops from w', using the following sequence and defining the mapping L:

$$s_0 = \sup\{j : w(T_j^{N-1}) = O\},$$

$$s_i = \sup\{j : w(T_j^{N-1}) = w(T_{s_{i-1}+1}^{N-1})\},\ i \geqq 1,$$

and

$$Lw' = [w(T_{s_0}^{N-1}), w(T_{s_1}^{N-1}), \ldots, w(T_{s_n}^{N-1})],$$

where $w(T_{s_n}^{N-1}) = a_N$ (Fig. 3(c)). We note here that $2^{-(N-1)}Lw' \in \hat{W}_1 \cup \hat{V}_1$.

3) Make a path by concatenation of n parts chosen from the original path ;

$$L_{N-1}w = [w_0, w_1, \ldots, w_{n-1}, a_N],$$

where

$$w_i = [w(T_{s_i}^{N-1}), w(T_{s_i}^{N-1}+1) \ldots, w(T_{s_{i+1}}^{N-1}-1)],\quad i = 0, \cdots, n-1.$$

By steps 1)–3), we have obtained $L_{N-1}w \in W_N \cup V_N$ with all 2^{N-1}-scale loops of w erased (Fig. 3(d)).

Using above as a base step, we shall now describe the induction step of our operation: Let $w \in W_N \cup V_N$. For $M \leqq N$, assume that all of the 2^N- to 2^M-scale loops have been erased from the path w, and denote the resulting path w', and its 2^M-skeleton by $\sigma_M(w')$. Additionally, for each $\Delta \in \sigma_M(w')$, we shall (implicitly) use the identification η defined in (2.1) to identify $Q_{M-1}w'|_\Delta$ with a path in $W_1 \cup V_1$.

L1) Coarse-grain w' to obtain $Q_{M-1}w'$ and consider

$$Q_{M-1}w'|_\Delta = [w'(T_k^{M-1}), w'(T_{k+1}^{M-1}), \ldots, w'(T_{k+k_0}^{M-1})],$$

where $w'(T_k^{M-1})$ is the entrance point to Δ and $w'(T_{k+k_0}^{M-1})$ the exit point from Δ.

L2) Erase loops from $Q_{M-1}w'|_\Delta$ as in the procedure for $W_1 \cup V_1$ by defining the sequence $\{s_i\}_{i=1}^n$ by

$$s_0 = \sup\{j : w'(T_j^{M-1}) = w'(T_k^{M-1})\},$$

$$s_i = \sup\{j : w'(T_j^{M-1}) = w'(T_{s_{i-1}+1}^{M-1})\},\ i \geqq 1,$$

and denoting

$$L(Q_{M-1}w'|_\Delta) = [w'(T_{s_0}^{M-1}), w'(T_{s_1}^{M-1}), \ldots, w'(T_{s_n}^{M-1})],$$

where $w'(T_{s_0}^{M-1}) = w'(T_k^{M-1})$ and $w'(T_{s_n}^{M-1}) = w'(T_{k+k_0}^{M-1})$.

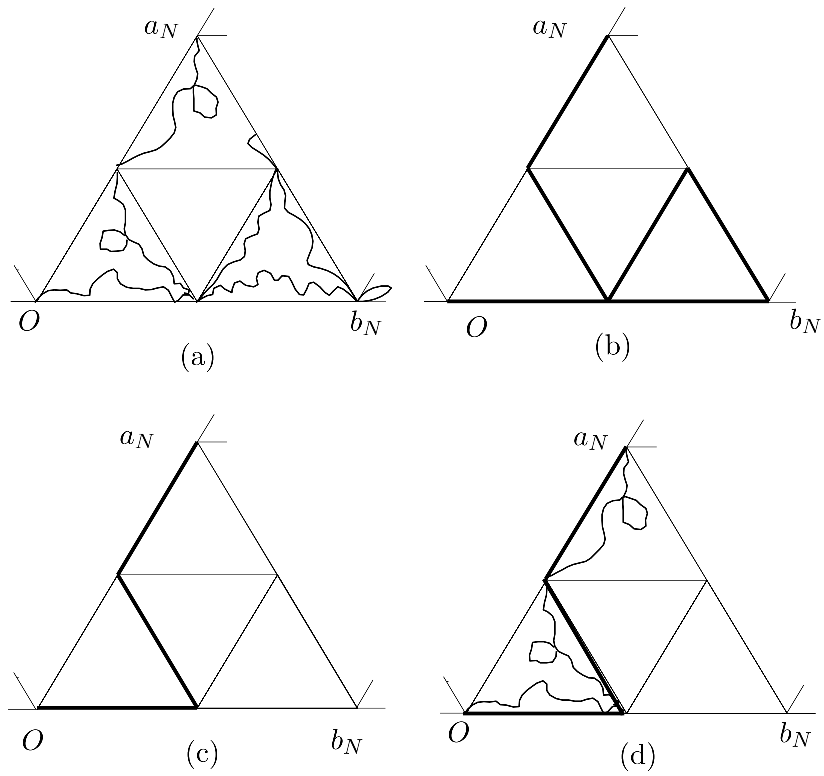

Fig 3: The loop-erasing procedure: (a) w, (b) w', (c) Lw', (d) $L_{N-1}w$

L3) Make a path segment in Δ by concatenation of n parts chosen from the original path and the exit point and denote it by

$$L_{M-1}(w|_\Delta) = [w'_0, w'_1, \ldots, w'_{n-1}, w'(T^{M-1}_{s_n})],$$

where

$$w'_i = [w'(T^{M-1}_{s_i}), w'(T^{M-1}_{s_i}+1)\ldots, w'(T^{M-1}_{s_{i}+1}-1)], \quad i = 0, \cdots, n-1.$$

L4) Make a whole path $w'' = L_{M-1}w$ by concatenation of parts obtained in L3) over all $\Delta \in \sigma_M(w')$.

Thus, by the procedure above, we have erased all of the 2^{M-1}-scale loops from w. Now denote by $\hat{Q}_{M-1}w$ the path obtained by concatenation of $L(Q_{M-1}w'|_\Delta)$ obtained in L2) over all $\Delta \in \sigma_M(w')$; then it is a path on F_{M-1}, in the sense that $Q_{M-1}(\hat{Q}_{M-1}w) = \hat{Q}_{M-1}w$, from O to a_N without loops. Observe that $\hat{Q}_{M-1}w = Q_{M-1}w''$. Although it may occur that $\sigma_{M-1}(w'') \neq \sigma_{M-1}(w')$, it holds that $\sigma_M(w'') = \sigma_M(w')$, which can be extended to $\sigma_K(w') = \sigma_K(w'')$ for any $K \geqq M$.

We then continue this operation until we have erased all of the loops and have $Lw = L_0 w = \hat{Q}_0 w$. Thus, by construction, our loop-erasing operation is essentially a repetition of loop-erasing for $W_1 \cup V_1$. We remark that the procedure implies that for any $w \in W_N \cup V_N$,

$$\sigma_K(\hat{Q}_M w) = \sigma_K(\hat{Q}_K w) \ \text{ for any } M \leqq K \leqq N. \tag{2.2}$$

In particular,

$$\sigma_K(Lw) = \sigma_K(\hat{Q}_K w) \ \text{ for } K \leqq N. \tag{2.3}$$

i.e., in the process of loop-erasing, once loops of 2^K-scale and greater have been erased, the 2^K-skeleton does not change any more. However it should be noted that the types of the triangles can change from Type 2 to Type 1.

2.4 Loop-erased random walks on the pre-Sierpiński gaskets.

Let $(\tilde{\Omega}, \mathcal{F}, P)$ be a probability space. A simple random walk on F_0 is a G_0-valued Markov chain $\{Z(i) : i \in \mathbb{Z}_+\}$ with transition probabilities

$$P[Z(i+1) = y \mid Z(i) = x] = \begin{cases} \frac{1}{4} & \text{if } y \in \mathcal{N}_0(x) \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathcal{N}_0(x)$ is defined in 2.1. Throughout this paper, we will consider random walks starting at O, so finite random walk paths are elements of W^*, and thus, T_i^N's and $Q_N Z$ can be defined.

Consider two kinds of random walks stopped at a_N: one conditioned on $Z(T_1^N) = a_N$ (before hitting other G_N vertices), called X_N, and the other conditioned on $Z(T_1^N) = b_N$ and $Z(T_2^N) = a_N$, i.e. hitting b_N on the way to a_N, called X'_N. These random walks then induce measures P_N and P'_N on W^* with support on W_N and V_N, respectively, namely, for $w \in W_N$,

$$\begin{aligned} P_N[w] &= P[X_N(i) = w(i), i = 0, 1, \ldots, \ell(w)] \\ &= P[Z(i) = w(i), i = 0, 1, \ldots, \ell(w) | Z(T_1^N) = a_N], \end{aligned}$$

and for $w \in V_N$,

$$\begin{aligned} P'_N[w] &= P[X'_N(i) = w(i), i = 0, 1, \ldots, \ell(w)] \\ &= P[Z(i) = w(i), i = 0, 1, \ldots, \ell(w) | Z(T_1^N) = b_N, \quad Z(T_2^N) = a_N]. \end{aligned}$$

Note that by symmetry:

$$P[Z(T_1^N) = a_N] = 1/4, \quad P[Z(T_1^N) = b_N, \quad Z(T_2^N) = a_N] = 1/16.$$

For the rest of this paper, the following propositions on the simple random walks on the pre-Sierpiński gasket will be used. They are straightforward consequences of the 'self-similarity', that is, $2^{-M} F_M = F_0$, and the property that if $x_0 \in G_M$ for some $M \in \mathbb{Z}_+$, then for each $x \in \mathcal{N}_M(x_0)$

$$P[Z(T_{i+1}^M) = x | Z(T_i^M) = x_0] = \frac{1}{4}$$

holds. (For details of random walks on the Sierpiński gasket, we refer to [2].)

Proposition 1 *If $M \leqq N$, then the distributions of $2^{-M} Q_M X_N$ and $2^{-M} Q_M X'_N$ are equal to P_{N-M} and P'_{N-M}, respectively; in other words, $Q_M X_N$ and $Q_M X'_N$ are simple random walks on a coarse graph F_M stopped at a_N and appropriately conditioned.*

Let η be the identification map defined in (2.1).

Proposition 2 *Let $M \leqq N$, and consider random walk segments conditioned on $Q_M X_N$ between the hitting times,*

$$Z_i = [X_N(t); \; T_i^M(X_N) \leqq t \leqq T_{i+1}^M(X_N)], \quad i = 1, \ldots, m,$$

where $X_N(T_m^M) = a_N$. Then Z_i, $i = 1, \ldots, m$, when identified with paths in W_{N-M} by appropriate translation, rotation and reflection, are independent and have the same distribution as X_{N-M}.

By applying loop-erasing operation to random walks X_N and X'_N, we induce measures $\hat{P}_N = P_N \circ L^{-1}$ supported on $\hat{W}_N$, and $\hat{P}'_N = P'_N \circ L^{-1}$ supported on $\hat{W}_N \cup \hat{V}_N$, respectively. Paths in $\hat{W}_1$ and $\hat{V}_1$ are shown in Fig. 4.

Their probabilities under $\hat{P}_1$ and $\hat{P}'_1$, respectively, can be obtained by direct calculation:

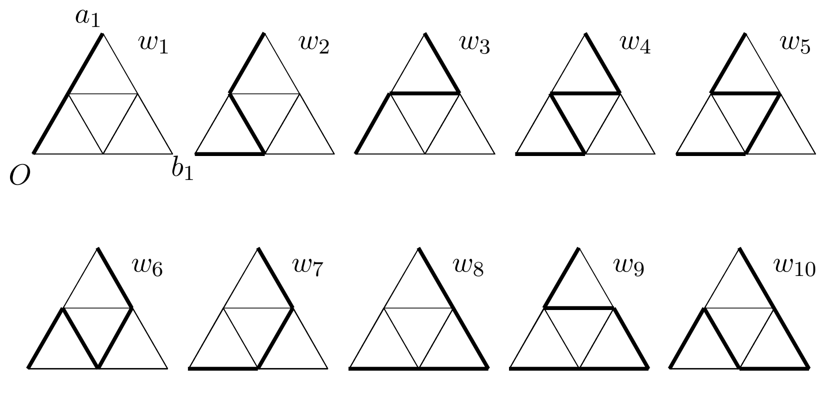

Fig 4: Paths in $\hat{W}_1 \cup \hat{V}_1$

$$\hat{P}_1[w_1] = \frac{1}{2}, \hat{P}_1[w_2] = \frac{2}{15}, \hat{P}_1[w_3] = \frac{2}{15}, \hat{P}_1[w_4] = \frac{1}{30}, \hat{P}_1[w_5] = \frac{1}{30},$$

$$\hat{P}_1[w_6] = \frac{1}{30}, \hat{P}_1[w_7] = \frac{2}{15}, \hat{P}_1[w_i] = 0,\ i = 8, 9, 10,$$

$$\hat{P}'_1[w_1] = \frac{1}{9}, \hat{P}'_1[w_2] = \frac{11}{90}, \hat{P}'_1[w_3] = \frac{11}{90}, \hat{P}'_1[w_4] = \frac{2}{45}, \hat{P}'_1[w_5] = \frac{2}{45},$$

$$\hat{P}'_1[w_6] = \frac{2}{45}, \hat{P}'_1[w_7] = \frac{8}{45}, \hat{P}'_1[w_8] = \frac{2}{9}, \hat{P}'_1[w_9] = \frac{1}{18}, \hat{P}'_1[w_{10}] = \frac{1}{18}.$$

For $w \in \hat{W}_N \cup \hat{V}_N$, let us denote the number of Type 1 triangles and Type 2 triangles in $\sigma_0(w)$ by $s_1(w)$ and $s_2(w)$, respectively. (This implies that $\ell(w) = s_1(w) + 2s_2(w)$.) Define two sequences, $\{\Phi_N\}_{N\in\mathbb{N}}$ and $\{\Theta_N\}_{N\in\mathbb{N}}$, of generating functions by:

$$\Phi_N(x,y) = \sum_{w\in\hat{W}_N} \hat{P}_N(w) x^{s_1(w)} y^{s_2(w)},$$

$$\Theta_N(x,y) = \sum_{w\in\hat{V}_N} \hat{P}'_N(w) x^{s_1(w)} y^{s_2(w)}, \quad x, y \geqq 0.$$

For simplicity, we shall denote $\Phi_1(x,y)$ and $\Theta_1(x,y)$ by $\Phi(x,y)$ and $\Theta(x,y)$.

Proposition 3 *The above generating functions satisfy the following recursion relations for all $N \in \mathbb{N}$:*

$$\Phi(x,y) = \frac{1}{30}(15x^2 + 8xy + y^2 + 2x^2y + 4x^3).$$

$$\Theta(x,y) = \frac{1}{45}(5x^2 + 11xy + 2y^2 + 14x^2y + 8x^3 + 5xy^2).$$

$$\Phi_{N+1}(x,y) = \Phi_N(\Phi(x,y), \Theta(x,y)).$$

$$\Theta_{N+1}(x,y) = \Theta_N(\Phi(x,y), \Theta(x,y)).$$

Proof. We shall first express $\hat{P}_{N+1}$ in terms of $\hat{P}_N$, $\hat{P}_1$ and $\hat{P}'_1$. If we recall the procedure for obtaining $\hat{Q}_1 X_{N+1}$ from X_{N+1}, we notice that it is the same as the procedure to obtain LX_N from X_N, except that everything is twice larger in the case of X_{N+1}. This together with Proposition 1 implies that the distribution of $2^{-1}\hat{Q}_1 X_{N+1}$ is equal to $\hat{P}_N$, namely,

$$P_{N+1}[\ v\ :\ \frac{1}{2}\hat{Q}_1 v = u\ ] = \hat{P}_N[\ u\ ].$$

On the other hand, we have from (2.2)

$$\sigma_1(\hat{Q}_1 X_{N+1}) = \sigma_1(LX_{N+1}).$$

The rest of the loop-erasing procedure to obtain LX_{N+1} together with Proposition 2 implies that conditioned on $\hat{Q}_1 X_{N+1}$, the walk segments of $L_1 X_{N+1}$ in $\Delta \in \sigma_1(\hat{Q}_1 X_{N+1})$ have the same distribution as either X_1 or X_1' (modulo appropriate transformation), and that they are mutually independent, which further implies that $LX_{N+1}|_\Delta$ are independent.

Keeping these observations in mind, we calculate $\hat{P}_{N+1}[w]$ for $w \in \hat{W}_{N+1}$. Let $\sigma_1(w) = (\Delta_1, \ldots, \Delta_k)$ be the 2^1-skeleton of w and let $w_i = w|_{\Delta_i}$ and let ηw_i be their identification with paths in $W_1 \cup V_1$ as defined in (2.1). Let $\sum_u$ denote the sum taken over $u \in \hat{W}_N$ satisfying $\sigma_0(u) = \frac{1}{2}\sigma_1(w)$, which consists of $\Delta_1, \ldots, \Delta_k$ scaled by 1/2.

Thus, we have

$$\begin{aligned}
\hat{P}_{N+1}[\ w\ ] &= P_{N+1}[\ v\ :\ Lv = w\ ] \\
&= \sum_u P_{N+1}[\ Lv = w,\ \frac{1}{2}\hat{Q}_1 v = u\ ] \\
&= \sum_u P_{N+1}[\ Lv = w \mid \frac{1}{2}\hat{Q}_1 v = u\ ]\ P_{N+1}[\ \frac{1}{2}\hat{Q}_1 v = u\ ] \\
&= \sum_u P_{N+1}[\ Lv = w \mid \frac{1}{2}\hat{Q}_1 v = u\ ]\ \hat{P}_N[\ u\ ] \\
&= \sum_u P_{N+1}[\ \eta(Lv|_{\Delta_i}) = \eta w_i, i = 1, \ldots, k \mid \frac{1}{2}\hat{Q}_1 v = u\ ]\ \hat{P}_N[\ u\ ] \\
&= \sum_u (\prod_{i=1}^k \hat{P}_1^*[\ \eta w_i\ ])\ \hat{P}_N[\ u\ ],
\end{aligned}$$

where $\hat{P}_1^* = \hat{P}_1$ if Δ_i is of Type 1, and $\hat{P}_1^* = \hat{P}_1'$ if Δ_i is of Type 2.

Since taking the sum over $w \in \hat{W}_{N+1}$ means taking the sum over all $u \in \hat{W}_N$ and finer structures in each $\Delta \in \sigma_1(w)$, we have

$$\begin{aligned}
\Phi_{N+1}(x, y) &= \sum_{w \in \hat{W}_{N+1}} \hat{P}_{N+1}(w) x^{s_1(w)} y^{s_2(w)} \\
&= \sum_{u \in \hat{W}_N} \sum_{\eta w_1 \in \hat{W}_1^*} \cdots \sum_{\eta w_k \in \hat{W}_1^*} (\prod_{i=1}^k \hat{P}_1^*[\ \eta w_i\ ])\ \hat{P}_N[\ u\ ] \\
&\quad \times x^{s_1(w_1) + \cdots + s_1(w_k)} y^{s_2(w_1) + \cdots + s_2(w_k)} \\
&= \sum_{u \in \hat{W}_N} \hat{P}_N[\ u\ ] \prod_{i=1}^k (\sum_{w_i \in \hat{W}_1^*} \hat{P}_1^*[\ w_i\ ]\ x^{s_1(w_i)} y^{s_2(w_i)}\ ) \\
&= \sum_{u \in \hat{W}_N} \hat{P}_N[\ u\ ]\ \Phi(x, y)^{s_1(u)} \Theta(x, y)^{s_2(u)} \\
&= \Phi_N(\Phi(x, y), \Theta(x, y)).
\end{aligned}$$

The calculations for $\hat{P}_{N+1}'$ and $\Theta_{N+1}(x, y)$ are similar. $\square$

Define the mean matrix by

$$\mathbf{M} = \begin{bmatrix} \frac{\partial}{\partial x}\Phi(1,1) & \frac{\partial}{\partial y}\Phi(1,1) \\ \frac{\partial}{\partial x}\Theta(1,1) & \frac{\partial}{\partial y}\Theta(1,1) \end{bmatrix} = \begin{bmatrix} \frac{9}{5} & \frac{2}{5} \\ \frac{26}{15} & \frac{13}{15} \end{bmatrix}. \tag{2.4}$$

It is a strictly positive matrix, and the larger eigenvalue is

$$\lambda = \frac{1}{15}(20 + \sqrt{205}) = 2.2878\ldots.$$

The loop-erasing procedure together with Proposition 2 leads to

Proposition 4 *Let $M \leqq N$. Conditioned on $\sigma_M(LX_N) = (\Delta_1, \ldots, \Delta_k)$ and the types of each element of the skeleton, the traverse times of the triangles*

$$T_i^{ex,M}(LX_N) - T_{i-1}^{ex,M}(LX_N), \quad i = 1, 2, \ldots, k$$

are independent. Each of them has the same distribution as either $T_1^{ex,N-M}(LX_{N-M})$ or $T_1^{ex,N-M}(LX'_{N-M})$, according to whether Δ_i is of Type 1 or Type 2.

Theorem 5 *As $N \to \infty$, $\lambda^{-N}\ell(LX_N)$ converges in law to an integrable random variable W', with a positive probability density.*

We shall prove the above theorem in Section 4 in a stronger form, using coupling argument. Theorem 5 suggests that the displacement exponent for the loop-erased random walk on the pre-Sierpiński gasket is $\log 2/\log\lambda$, in the sense that the average number of steps it takes to cover the distance of 2^N is of order λ^N. In other words, if we write $m = 2^N$, it takes $m^{\log\lambda/\log 2}$ steps to travel a distance of m from the origin. This exponent is obtained in [5] and proved by Shinoda, using the uniform spanning tree.

3 Equivalence

In this section we show the equivalence of the LERW obtained by the 'erasing-larger-scale-loops-first' rule to the standard LERW obtained by erasing loops in chronological order. Erasing loops in a different order can result in a different path measure, but here we show that these two LERWs are equivalent.

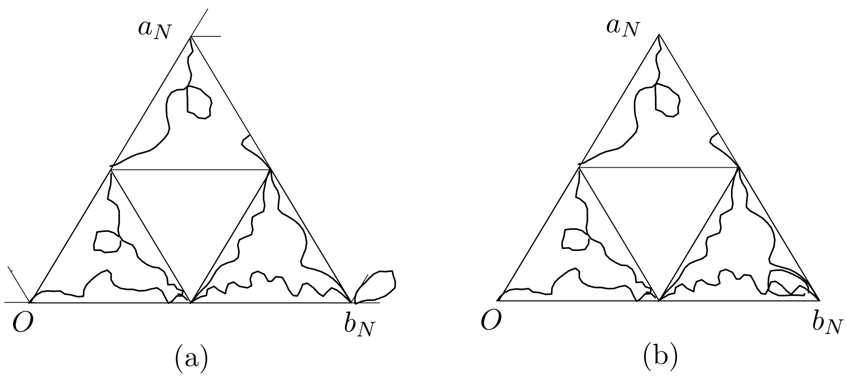

Fig 5: (a) X'_N on F_0, (b) $\tilde{X}_N$ on F'_N

In order to compare with the standard LERW, we need some preparation. First, we limit ourselves to the series of finite graphs $F'_N = F_0 \cap \triangle Oa_Nb_N$, $N \in \mathbb{Z}_+$ (Fig 1). Consider a simple random walk $\tilde{X}_N$ on F'_N starting at O and stopped at the first hitting time of a_N. Here we do not set any condition on visit to b_N. We shall construct LERW from $\tilde{X}_N$ according to the 'erasing-larger-scale-loops-first' rule and compare it with the standard LERW. Since $\tilde{X}_N$ may have 2^N-scale loops ($\tilde{X}_N$ may go back and forth between O and b_N), which X_N and X'_N in Section 2.4 did not have, we have to erase them first. As before, we start with coarse-graining the walk to get $Q_N\tilde{X}_N$, and then similarly to Proposition 1, we see that it is a simple random walk on $2^N F_0'$ (F_0' magnified by 2^N). We erase loops from $Q_N\tilde{X}_N$: First, we erase all the loops formed at O and progress one step. If the new position is a_N, loop-erasing is done. If the new position is b_N, erase all the loops there. Since the next step necessarily brings the walk to a_N, loop-erasing is done. Denote the resulting walk by $\hat{Q}_N\tilde{X}_N$ according to the notation introduced just below L4) in Section 2.3. There are two loopless paths connecting O and a_N on $2^N F_0'$, $\tilde{w}_1 = (O, a_N)$ and $\tilde{w}_2 = (O, b_N, a_N)$. By direct calculation we have

$$P[\ \hat{Q}_N\tilde{X}_N = \tilde{w}_1\ ] = \frac{2}{3},\ \ P[\ \hat{Q}_N\tilde{X}_N = \tilde{w}_2\ ] = \frac{1}{3}.$$

Then we proceed to give back the fine structures of the original walk to each step of $\tilde{w}_1$ or $\tilde{w}_2$ to obtain $L_N\tilde{X}_N$ as L3) and L4) in Section 2.3. Conditioned on $\hat{Q}_N\tilde{X}_N = \tilde{w}_1$, $L_N\tilde{X}_N$ does not hit b_N on the way to a_N, while conditioned on $\hat{Q}_N\tilde{X}_N = \tilde{w}_2$, $L_N\tilde{X}_N$ hit b_N in such a way as $\tilde{X}_N(T_1^N) = b_N$ and $\tilde{X}_N(T_2^N) = a_N$. This reminds us of the two kinds of random walks X_N and X_N' on F_0 stopped at a_N defined in Section 2.4. The only difference lies in that for X_N and X_N' we allowed loops to leak into neighboring triangles from O and b_N, as long as the diameters of the loops are smaller than 2^N. However, if we fold these loops inward at O and b_N, then we get random walks on F_N' (Fig 5). X_N with leaking loops folded back has the same law with $\tilde{X}_N$ conditioned on $\hat{Q}_N\tilde{X}_N = \tilde{w}_1$, and X_N' with loops folded the same law with $\tilde{X}_N$ conditioned on $\tilde{Q}_N\tilde{X}_N = \tilde{w}_2$. Now we can apply the loop-erasing procedure described in Section 2.3 to obtain a LERW measure $\tilde{P}_N$ on $\hat{W}_N \cup \hat{V}_N$. Since loop-folding does not affect the result of loop-erasing, we have

$$\tilde{P}_N = \frac{2}{3}\hat{P}_N + \frac{1}{3}\hat{P}_N', \tag{3.1}$$

where $\hat{P}_N$ and $\hat{P}_N'$ are defined in Section 2.4.

We shall compare $\tilde{P}_N$ with the standard LERW (obtained by chronological loop-erasure). To this end, we shall review briefly the relation between the standard LERW and the uniform spanning tree. Consider a finite connected graph $G = (V, E)$, where V denotes the set of vertices and E the set of edges. A **spanning forest** on G is a subgraph $T = (V', E')$ of G such that $V' = V$ and T has no cycles (loops). A connected spanning forest is called a **spanning tree**. If we assign a uniform measure to the set of all the spanning trees on G, we have the **uniform spanning tree**. For each spanning tree T and for any pair of vertices $v, w \in V$, there exists a unique loopless path connecting them on T. Pemantle proved that the induced path measure coincides with the standard LERW path measure on G conditioned on starting at v and stopped at w [19].

Now we go back to the pre-Sierpiński gasket. In order to relate LERW measure on F_{N+1}' to that on F_N', we need both spanning trees and spanning forests. Let $\mathcal{A}_N$ be the set of all spanning trees on F_N' and denote the uniform measure on it by P_N^{UST}. Let $\mathcal{A}_N'$ be the set of all spanning forests on F_N' consisting of two connected parts, one containing O and a_N and the other b_N and denote the uniform measure on $\mathcal{A}_N'$ by P_N^{USF}.

For each $T \in \mathcal{A}_N \cup \mathcal{A}_N'$, there exists a unique loopless path that connects O and a_N. Let us denote this mapping by

$$H_N : \mathcal{A}_N \cup \mathcal{A}_N' \longrightarrow \hat{W}_N \cup \hat{V}_N.$$

Denote the induced measures by $\hat{P}_N^{UST} = P_N^{UST} \circ H_N^{-1}$ and $\hat{P}_N^{USF} = P_N^{USF} \circ H_N^{-1}$ respectively. As stated above, $\hat{P}_N^{UST}$ coincides with the standard LERW measure on F_N'. Furthermore, we can show

Theorem 6

$$\hat{P}_N^{UST} = \tilde{P}_N,\ \ \hat{P}_N^{USF} = \hat{P}_N. \tag{3.2}$$

Proof. We prove the theorem by induction on N. By direct calculation, we obtain the second equality for $N=1$, and

$$\hat{P}_1^{UST}[w_1] = \frac{20}{54},\ \hat{P}_1^{UST}[w_2] = \hat{P}_1^{UST}[w_3] = \frac{7}{54},$$
$$\hat{P}_1^{UST}[w_4] = \hat{P}_1^{UST}[w_5] = \hat{P}_1^{UST}[w_6] = \frac{2}{54},$$
$$\hat{P}_1^{UST}[w_7] = \frac{8}{54},\ \hat{P}_1^{UST}[w_8] = \frac{4}{54},\ \hat{P}_1^{UST}[w_9] = \hat{P}_1^{UST}[w_{10}] = \frac{1}{54},$$

where $w_1, \dots, w_{10}$ are shown in Fig 4. If we recall $\tilde{P}_1 = \frac{2}{3}\hat{P}_1 + \frac{1}{3}\hat{P}'_1$, we see the first equality holds for $N=1$.

Suppose that (3.2) holds for N. For any $w \in \hat{W}_{N+1} \cup \hat{V}_{N+1}$, we will show that $\hat{P}_{N+1}^{UST}[w] = \tilde{P}_{N+1}[w]$.

Let $\Delta_O, \Delta_a, \Delta_b$ be the 2^N-triangles in $\triangle O a_{N+1} b_{N+1}$ containing O, a_{N+1}, b_{N+1}, respectively. Recall the 2^N-skeleton $\sigma_N(w)$ of the path $w \in \hat{W}_{N+1} \cup \hat{V}_{N+1}$ from Section 2.3. $\sigma_N(w)$ is either (Δ_O, Δ_a) or $(\Delta_O, \Delta_b, \Delta_a)$. Let us prove in the case that $\sigma_N(w) = (\Delta_O, \Delta_a)$, that is, w passes through only two 2^N-triangles. The proof in the case of $\sigma_N(w) = (\Delta_O, \Delta_b, \Delta_a)$ is the same. Divide the path into the part in Δ_O and that in Δ_a, and regard them as paths on F'_N (modulo appropriate rotation and reflection), $w_1 = w|_{\Delta_O}$ and $w_2 = w|_{\Delta_a}$.

First consider $\hat{P}_{N+1}^{UST}[w]$. Let $T \in \mathcal{A}_{N+1}$ and consider its three parts $T \cap \Delta_O, T \cap \Delta_a$ and $T \cap \Delta_b$. Two of them should be congruent to some spanning trees in $\mathcal{A}_N$ and the rest congruent to some $T' \in \mathcal{A}'_N$. (Since there should be no cycles, all three cannot be spanning trees.) For example, if $T \cap \Delta_b$ is congruent to some $T' \in \mathcal{A}'_N$, then the forest should be embedded in Δ_b so that a_N and b_N belong to distinct connected components. (Otherwise, we have a cycle.) There are two possible rotations of T' allowed. Thus, we can count six kinds of possible configurations of two trees and a forest, taking rotation of the forest into account, and they occur with equal probability. For example, the probability that $T \cap \Delta_O$ and $T \cap \Delta_a$ are congruent to some trees in $\mathcal{A}_N$ and that $T \cap \Delta_b$ connects b_N and b_{N+1} and is congruent to some forest in $\mathcal{A}'_N$ is 1/6. (For more details, see [4].) Since parts of the path in distinct 2^N-triangles are independent conditioned on $\sigma_N(w)$, we have

$$\begin{aligned}\hat{P}_{N+1}^{UST}[\quad w \quad]& \\ =\ & \frac{1}{3}\hat{P}_N^{UST}[w_1]\hat{P}_N^{UST}[w_2] + \frac{1}{6}\hat{P}_N^{UST}[w_1]\hat{P}_N^{USF}[w_2] + \frac{1}{6}\hat{P}_N^{USF}[w_1]\hat{P}_N^{UST}[w_2].\end{aligned}$$

Now consider $\tilde{P}_{N+1}[w]$. We recall the procedure by the 'erasing-larger-scale-loops-first' rule, and notice that for $w \in \hat{W}_{N+1} \cup \hat{V}_{N+1}$, if $v \in W_{N+1}$ satisfies $Lv = w$, then by (2.3), $\sigma_N(\hat{Q}_N v) = \sigma_N(w) = (\Delta_O, \Delta_a)$. Thus we classify simple random walk paths according to $\hat{Q}_N \tilde{X}_N$ and using $P_{N+1}[2^{-N}\hat{Q}_N v = u] = \hat{P}_1[u]$, we have

$$\begin{aligned}\hat{P}_{N+1}[w] &= P_{N+1}[v : Lv = w] = \sum_u P_{N+1}[\ Lv = w,\ 2^{-N}\hat{Q}_N v = u\ ] \\ &= \sum_u P_N[\ Lv|_{\Delta_O} = w_1,\ Lv|_{\Delta_a} = w_2 \mid 2^{-N}\hat{Q}_N v = u\ ]\hat{P}_1[u] \\ &= \sum_u \hat{P}_N^*[w_1]\hat{P}_N^*[w_2]\hat{P}_1[u],\end{aligned}$$

where $\sum_u$ is taken over the paths in $\hat{W}_1 \cup \hat{V}_1$ such that $2^N\sigma_0(u) = (\Delta_O, \Delta_a)$, and $\hat{P}_N^*[w_i] = \hat{P}_N$ if the i-th element of $\sigma_0(u)$ is of Type 1, and $\hat{P}_N^* = \hat{P}'_N$ if of Type 2. We have a similar result also for $\hat{P}'_{N+1}[w]$. These combined with (3.1) lead to

$$\tilde{P}'_{N+1}[w] = \sum_u \hat{P}_N^*[w_1]\hat{P}_N^*[w_2](\frac{2}{3}\hat{P}'_1[u] + \frac{1}{3}\hat{P}'_1[u]).$$

Substituting the explicit values of $\hat{P}_1$ and $\hat{P}_1'$ given in Section 2.4 into the right-hand side and then using the induction assumption, we have

$$\hat{P}^{UST}_{N+1}[w] = \tilde{P}_{N+1}[w].$$

The second equality is proved similarly. $\square$

4 Scaling limit of the loop-erased random walks.

4.1 Paths on the Sierpiński gasket.

In this section we investigate the limit of the loop-erased random walk as the lattice spacing (edge length) tends to 0. First we define the (finite) Sierpiński gasket. Since it will be easier to deal with continuous functions from the beginning, we regard F_0 as a closed subset of $\mathbb{R}^2$ made up of all the points on its edges. Let Δ_1 be the closed (filled) triangle in $\mathcal{T}_0$ whose vertices are O, a_0 and b_0, and Δ_2 be its reflection with regard to the y-axis, and let $F^N = 2^{-N} F_0 \cap (\Delta_1 \cup \Delta_2)$ (Fig 6). We define the **Sierpiński gasket** by $F = cl(\cup_{N=0}^{\infty} F^N)$, where cl denotes closure. We define the sets of vertices on F by $G^N = 2^{-N} G_0 \cap (\Delta_1 \cup \Delta_2)$.

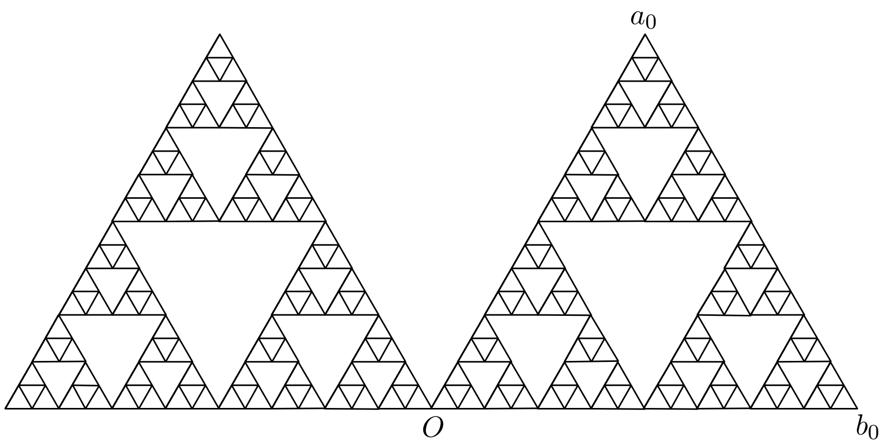

Fig 6: F^N

Let

$$C = \{w \in C([0,\infty) \to F) \ : \ w(0) = O, \ \lim_{t\to\infty} w(t) = a_0\}.$$

C is a complete separable metric space with the metric

$$d(u,v) = \sup_{t\in[0,\infty)} |u(t) - v(t)|, \ u, v \in C,$$

where $|x-y|$, $x, y \in \mathbb{R}^2$, denotes the Euclidean distance. Throughout this section, for $w \in \bigcup_{N=1}^{\infty} W_N$, we let

$$w(t) = a_N, \quad t \geqq \ell(w),$$

and interpolate the paths linearly,

$$w(t) = (i+1-t)w(i) + (t-i)w(i+1), \quad i \leqq t < i+1, \quad i = 0, 1, 2, \cdots$$

so that we can regard w as a continuous function on $[0,\infty)$. Hereafter we assume that all paths are linearly interpolated. Let

$$W^N = 2^{-N} W_N = \{2^{-N} w \ : \ w \in W_N\}, \quad \hat{W}^N = 2^{-N} \hat{W}_N.$$

Thus, W^N and $\hat{W}^N$ are subsets of C. For $w \in W^N$, let $\tilde{\ell}(w) = \ell(2^N w)$. Namely, $\tilde{\ell}(w)$ is the number of 2^{-N}-sized 'steps' the path w takes to get to a_0.

We define hitting times, coarse-graining, exit times and skeletons similarly to Section 2, but with G_M replaced by G^M. Namely, for $w \in C$ we define a sequence $\{T_i^M(w)\}_{i=0}^m$ of the hitting times of G^M, as follows: $T_0^M(w) = 0$, and for $i \geqq 1$, let $T_i^M(w) = \inf\{j > T_{i-1}^M(w) \ : \ w(j) \in G^M \setminus \{w(T_{i-1}^M(w))\}\}$. m is the smallest integer such that $T_{m+1}^M(w) = \infty$. For the hitting times we are using the same notation but we hope no confusion arises. For $N \in \mathbb{Z}_+$, we define a coarse-graining map $Q^N : C \to C$ by $(Q^N w)(i) = w(T_i^N(w))$ for $i = 0, 1, 2, \ldots, m$, and by using linear interpolation

$$(Q^N w)(t) = \begin{cases} (i+1-t)\ (Q^N w)(i) & +(t-i)\ (Q^N w)(i+1), \\ & i \leqq t < i+1,\ i = 0,1,2,\ldots,m-1, \\ a_0, & t \geqq m. \end{cases}$$

Notice that

$$Q^M \circ Q^N = Q^M, \quad \text{ if } \quad M \leqq N \tag{4.1}$$

holds.

Since we have defined the hitting times for every $w \in C$, we can define its 2^{-M}-**skeleton**, $\sigma^M(w)$ (a sequence of 2^{-M}-triangles w passes through) and the **exit times** $\{T_i^{ex,M}\}$ similarly to their counterparts in Section 2. To define the loop erasing operator, recall that if $w \in W^N$, then $2^N w \in W_N$ and $L(2^N w) \in \hat{W}_N$ (modulo linear interpolation). Thus we define loop erasure $\tilde{L} : \bigcup_{N=0}^\infty W^N \to \bigcup_{N=0}^\infty \hat{W}^N$ by letting $\tilde{L} w = 2^{-N} L(2^N w) \in \hat{W}^N$ for $w \in W^N$, $N \in \mathbb{Z}_+$, and we define also $\hat{Q}^M w = 2^{-N} \hat{Q}_M(2^N w) \in \hat{W}^M$ for $M \leqq N$. The only differences from the previous section are that paths are continuous (by linear interpolation) and confined in two neighboring unit triangles, and that we erase loops from 2^{-1}-scale down. For each $N \in \mathbb{Z}_+$, let P^N be the random walk path measure on F^N (a probability measure on C supported on W^N), namely $P^N[w] = P_N[2^N w]$, for $w \in W^N$. In the following, we will focus on P^N.

4.2 The scaling limit.

We consider random walks (linearly interpolated version) on G^N, $N \in \mathbb{Z}_+$, starting at O and stopped at a_0.

Let

$$\Omega' = \{\omega = (w_0, w_1, w_2, \cdots) \ : \ w_0 \in \hat{W}^0,\ w_N \in \hat{W}^N,\ w_N \triangleright w_{N+1},\ N \in \mathbb{N}\},$$

where $w_N \triangleright w_{N+1}$ means that there exists a $v \in W^{N+1}$ such that $Q^N v = w_N$ and $\hat{Q}^{N+1} v = w_{N+1}$. Define the projection onto the first $N+1$ elements by

$$\pi_N \omega = (w_0, w_1, \ldots, w_N),$$

and a probability measure on $\pi_N \Omega'$ by

$$\hat{P}^N[(w_0, w_1, \ldots, w_N)] = P^N[\ v : \ \hat{Q}^i v = w_i, i = 0, \ldots, N\ ].$$

The following consistency condition is a direct consequence of the loop-erasing procedure:

$$\hat{P}^N[(w_0, w_1, \ldots, w_N)] = \sum_u \hat{P}^{N+1}[(w_0, w_1, \ldots, w_N, u)], \tag{4.2}$$

where the sum is taken over all possible $u \in \hat{W}_{N+1}$ such that $w_N \triangleright u$.

By virtue of (4.2) and Kolmogorov's extension theorem for a projective limit, there is a probability measure $\hat{P}$ on $\Omega_0 = C^{\mathbb{N}} = C \times C \times \cdots$ such that

$$\hat{P}[\ \Omega'\ ] = 1.$$

$$\hat{P} \circ \pi_N^{-1} = \hat{P}^N,\ N \in \mathbb{Z}_+.$$

Let $Y^N:\ \Omega' \to C$ be the projection to the N-th component. We regard Y^N as an F-valued process $Y^N(\omega, t)$ on $(\Omega_0, \mathcal{B}, \hat{P})$, where $\mathcal{B}$ is the Borel algebra on Ω_0 generated by the cylinder sets.

For $w \in C$ and $j = 1, 2$, denote by $S_j^M(w)$ the number of 2^{-M}-triangles of Type j in $\sigma^M(w)$, namely, $S_j^M(w) = \sharp\{i:\ \Delta_i \text{ is of Type } j\ \}$, and let $\mathbf{S}^M(w) = (S_1^M(w), S_2^M(w))$. If $w \in W^N$ for some N, then $\tilde{\ell}(w) = S_1^N(w) + 2S_2^N(w)$.

Let $\mathbf{S} = (S_1, S_2)$ and $\mathbf{S}' = (S_1', S_2')$ be $\mathbb{Z}_+$-valued random variables on $(\Omega_0, \mathcal{B}, \hat{P})$ with the same distributions as those of (s_1, s_2) under $\hat{P}_1$ and under $\hat{P}_1'$, respectively. (s_1, s_2) has been defined in Section 2.4 together with the generating functions.

Proposition 7 *Fix arbitrarily $v \in \hat{W}^M$, and let $\sigma^M(v) = (\Delta_1, \ldots, \Delta_k)$. For each i, $1 \leqq i \leqq k$, under the conditional probability $\hat{P}[\ \cdot\ |Y^M = v]$, $\{\mathbf{S}^{M+N}(Y^{M+N}|_{\Delta_i}),\ N = 0, 1, 2, \cdots\}$ is a two-type supercritical branching process, with the types of children corresponding to the types of triangles. The offspring distributions born from a Type 1 triangle and from a Type 2 triangle are equal to those of $\mathbf{S}$ and $\mathbf{S}'$, respectively. If Δ_i is of Type 1, the process initiates in state $(1, 0)$, and if Δ_i is of Type 2, in state $(0, 1)$.*

(1) The generating functions for the offspring distributions are

$$g_1(x, y) \stackrel{\text{def}}{=} \hat{E}[\ x^{S_1} y^{S_2}\ ] = \Phi(x, y),$$

$$g_2(x, y) \stackrel{\text{def}}{=} \hat{E}[\ x^{S_1'} y^{S_2'}\ ] = \Theta(x, y),$$

where $\hat{E}$ is an expectation with regard to $\hat{P}$.

(2) The mean matrix $\mathbf{M}$ is given by (2.4) in Section 2. It is strictly positive and its eigenvalues are $\lambda = \frac{1}{15}(20 + \sqrt{205}) = 2.2878\ldots$ and $\lambda' = \frac{1}{15}(20 - \sqrt{205}) = 0.3788\ldots$. We have

$$\hat{E}[\ \mathbf{S}^{M+N}(Y^{M+N}|_{\Delta_i})\ |\ Y^M = v\ ] = \mathbf{S}^M(v|_{\Delta_i})\mathbf{M}^N.$$

(3) $\hat{P}[S_1 + S_2 \geqq 2] = \hat{P}[S_1' + S_2' \geqq 2] = 1$ (non-singularity).

(4)

$$\hat{E}[\ S_i \log S_i\ ] < \infty,\quad \hat{E}[\ S_i' \log S_i'\ ] < \infty,\ i = 1, 2.$$

Proposition 7 suggests that we should consider F-valued processes with time appropriately scaled. Thus, we introduce a time-scale transformation $U_N(\alpha): C \to C,\ \alpha \in (0, \infty),\ n \in \mathbb{N}$. For $w \in C$, define

$$(U_N(\alpha)w)(t) = w(\alpha^N t),$$

and consider the processes

$$X^N = U_N(\lambda)Y^N,\quad N \in \mathbb{Z}_+.$$

Proposition 8

$$\sigma^M(X^N) = \sigma^M(X^M) = \sigma^M(Y^M),\quad M \leqq N,\quad a.s.$$

In particular,

$$X^N(T_i^{ex,M}(X^N)) = X^M(T_i^{ex,M}(X^M)) = Y^M(T_i^{ex,M}(Y^M)),\quad M \leqq N,\quad a.s. \tag{4.3}$$

Note that if $\sigma^M(X^N) = (\Delta_1, \cdots, \Delta_k)$, then

$$T_j^{ex,M}(X^N) = \lambda^{-N} \sum_{i=1}^{j} (S_1^N(X^N|\Delta_i) + 2S_2^N(X^N|\Delta_i)), \quad 1 \leqq j \leqq k.$$

Proposition 7 combined with the convergence theorem for supercritical branching processes (see [1], Chapter V) leads to the following proposition.

Let $\mathbf{u} = {}^t(u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be the right and left positive eigenvectors associated with λ such that $|\mathbf{u}| = |\mathbf{v}| = 1$.

Proposition 9 *Fix arbitrarily $v \in \hat{W}^M$, and let $\sigma^M(v) = (\Delta_1, \ldots, \Delta_k)$. For each i, $1 \leqq i \leqq k$, under the conditional probability $\hat{P}[\ \cdot\ |Y^M = v]$, we have the following:*

(1) For each $i \in \{1, \cdots, k\}$, $\{\lambda^{-(M+N)}\mathbf{S}^{M+N}(X^{M+N}|_{\Delta_i}),\ N = 0, 1, 2, \ldots\}$ converges a.s. as $N \to \infty$ to a $\mathbb{R}^2$-valued random variable $\mathbf{S}^{*M,i} = (S_1^{*M,i}, S_2^{*M,i})$.

(2) $\{\mathbf{S}^{*M,i},\ i = 1, \cdots, k\}$ are independent.

(3) There are random variables B_1 and B_2 such that $\mathbf{S}^{*M,i}$ is equal in distribution to $\lambda^{-M}B_1\mathbf{v}$ if Δ_i is of Type 1, and equal in distribution to $\lambda^{-M}B_2\mathbf{v}$ if Δ_i is of Type 2.

(4)

$$\hat{P}[B_i > 0] = 1, \quad \hat{E}[B_i] = u_i, \quad i = 1, 2.$$

B_1 and B_2 have strictly positive probability density functions.

(5) The Laplace transform of B_i, $i = 1, 2$

$$\phi_i(t) = \hat{E}[\exp(tB_i)]$$

are entire functions on $\mathbb{C}$ and are the unique solution to

$$\phi_1(\lambda t) = \Phi(\phi_1(t), \phi_2(t)), \quad \phi_2(\lambda t) = \Theta(\phi_1(t), \phi_2(t)), \ \phi_1(0) = \phi_2(0) = 1.$$

To be precise, (1)–(4) in Proposition 9 are the straightforward consequences of general limit theorems for multi-type superbranching processes (Theorem 1 and Theorem 2 in V.6 of [1]). $\hat{P}[B_i > 0] = 1$ is a consequence of Φ and Θ having no terms with degree smaller than 2. For the existence of the Laplace transform on the entire $\mathbb{C}$, we need careful study of the recursions. We omit the details here, since they are similar to those in [10].

Let $T_i^{*M} = \sum_{j=1}^{i}(S_1^{*M,j} + 2S_2^{*M,j})$. Then $\lim_{N\to\infty} T_j^{ex,M}(X^N) = T_j^{*M}$. By virtue of Proposition 8 and Proposition 9, we can prove the almost sure uniform convergence for X^N.

Theorem 10 *X^N converges uniformly in t a.s. as $N \to \infty$ to a continuous process X.*

Proof. Choose $\omega \in \Omega'$ such that the following holds for all $M \in \mathbb{Z}_+$: $Y^M \in \hat{W}^M$, $\lim_{N\to\infty} T_i^{ex,M}(X^N) = T_i^{*M}$ exists and $T_i^{*M} - T_{i-1}^{*M} > 0$ for all $1 \leqq i \leqq k_M$, where k_M denotes the number of triangles in $\sigma^M(Y^M)$. Let $R = T_1^{*0} + \varepsilon$, where $\varepsilon > 0$ is arbitrary. It suffices to show that $X^N(\omega, t)$ converges uniformly in $t \in [0, R]$. In fact, if $t > R$, $X^N(t) = a_0$ for a large enough N.

Fix $M \geqq 0$. Let $k = k_M$. By expressing the arrival time at a_0 as the sum of traversing times of 2^{-M}-triangles, we have $T_k^{ex,M}(X^N) = T_1^{ex,0}(X^N)$ a.s. Letting $N \to \infty$, we have $T_k^{*M} = T_1^{*0}$ a.s.

The choice of ω shows that there exists an $N_1 = N_1(\omega) \in \mathbb{N}$ such that

$$\max_{1\leqq i\leqq k} |T_i^{ex,M}(X^N) - T_i^{*M}| \leqq \min_{1\leqq i\leqq k}(T_i^{*M} - T_{i-1}^{*M}), \tag{4.4}$$

and

$$|T_k^{ex,M}(X^N) - T_k^{*M}| < \varepsilon,$$

for $N \geqq N_1$.

If $0 \leqq t < T_k^{*M}$, then choose $j \in \{1, \cdots, k\}$ such that $T_{j-1}^{*M} \leqq t < T_j^{*M}$.

Then (4.4) implies that $T_{j-2}^{ex,M}(X^N) \leqq t \leqq T_{j+1}^{ex,M}(X^N)$, for $N \geqq N_1$. Since Proposition 8 shows

$$X^N(T_j^{ex,M}(X^N)) = X^M(T_j^{ex,M}(X^M)), \tag{4.5}$$

for all N with $N \geqq M$, we have

$$|X^N(T_j^{ex,M}(X^N)) - X^N(t)| \leqq 3 \cdot 2^{-M}.$$

Otherwise, if $T_k^{*M} \leqq t \leqq T_k^{*M} + \varepsilon = R$, then let $j = k$. Since $T_{k-1}^{ex,M}(X^N) \leqq t$,

$$|X^N(T_j^{ex,M}(X^N)) - X^N(t)| \leqq 2 \cdot 2^{-M}.$$

Therefore, if $N, N' \geqq N_1$, then for any $t \in [0, R]$,

$$\begin{aligned}
&|X^N(t) - X^{N'}(t)| \\
\leqq\ & |X^N(T_j^{ex,M}(X^N)) - X^N(t)| + |X^{N'}(T_j^{ex,M}(X^{N'})) - X^{N'}(t)| \\
&+|X^N(T_j^{ex,M}(X^N)) - X^{N'}(T_j^{ex,M}(X^{N'}))| \\
\leqq\ & 6 \cdot 2^{-M},
\end{aligned}$$

where the third term in the middle part is shown to be 0 by (4.5). Since M is arbitrary, we have the uniform convergence.

□

Theorem 11 *X is almost surely self-avoiding. The Hausdorff dimension of the path $X([0,\infty))$ is almost surely equal to* $\log \lambda / \log 2$.

The uniform convergence of X^N, which is self-avoiding, to X implies that the probability of the event that there exist t_1, t_2 and t_3 with $t_1 < t_2 < t_3$ such that $X(t_1) = X(t_3)$, $X(t_2) \neq X(t_1)$ is zero, and the existence of the Laplace transforms $\hat{E}[\exp(t_0 B_i)]$, $i = 1, 2$ for some $t_0 > 0$ guarantees that the probability that there exist $t_1, t_2 > 0$ such that $X(t) = X(t_1)$ for all t, $t_1 \leqq t \leqq t_1 + t_2$ is zero. We omit the detailed proof here since they are similar to that in [8]. To calculate the Hausdorff dimension, we use the fact that if a path w is loopless, then it holds that

$$\sigma^1(w) \supset \sigma^2(w) \supset \sigma^3(w) \supset \cdots \to w,$$

in the Hausdorff metric. Thus, we can regard the path as a multi-type random fractal to obtain the Hausdorff dimension in the same way as [11]. □

Since $\lambda^{-N}\ell(LX_N)$ in Theorem 5 has the same distribution as λ^{-N} $(S_1^N(X^N) + 2S_2^N(X^N))$, Theorem 5 follows immediately from Proposition 9, with W' equal in distribution to $(v_1 + 2v_2)B_1$.

5 Conclusion

We proposed an alternative procedure of loop-erasing from simple random walk on the finite pre-Sierpiński gasket. It is based on an 'erasing-larger-scale-loops-first' rule, which enables us to obtain exact recursion relations, without using the uniform spanning tree. First, we proved that the LERW above is equivalent to the standard LERW. Then we proved the existence of the scaling limit. We made use of the tools that have been developed for the study of self-avoiding walks on the pre-Sierpiński gasket to prove that the path of the limiting process is almost surely self-avoiding, while having Hausdorff dimension strictly greater than 1.

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