Displacement exponent for loop-erased random walk on the Sierpiński gasket

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ABSTRACT

We prove that loop-erased random walks on the finite pre-Sierpiński gaskets can be extended to a loop-erased random walk on the infinite pre-Sierpiński gasket by using the 'erasing-larger-loops-first' method, and obtain the asymptotic behavior of the walk as the number of steps increases, in particular, the displacement exponent and a law of the iterated logarithm.

Key words: loop-erased random walk ; displacement exponent ; growth exponent ; law of the iterated logarithm ; Sierpiński gasket ; fractal

MSC2010 Subject Classifications: 60F99, 60G17, 28A80, 37F25, 37F35

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1 Introduction

Loop-erased random walk (LERW) is a process obtained by erasing loops from a simple random walk in chronological order. It was first introduced on $\mathbb{Z}^d$ by G. Lawler ([10]) with the hope that it would give some perspective to the study of the self-avoiding walk (SAW). Although it turned out that LERW and SAW are in different universality classes, LERW has been attracting attention, in particular, because of the close relation to the uniform spanning tree. Two natural questions concerning the LERW are the existence of the scaling limit and the asymptotic behavior as the number of the steps tends to infinity. On $\mathbb{Z}^d$, the existence of the scaling limit has been proved for all d and the asymptotic behavior of the walk has been studied in terms of the growth exponent (expected to be the reciprocal of the displacement exponent). For the scaling limit of LERW on $\mathbb{Z}^d$, see, for example, [14] and [15] for $d = 2$, [9] for $d = 3$, [11] and [12] for $d \geqq 4$. For the growth exponents for LERW on $\mathbb{Z}^d$, see, for example, [8], [16] and [13] for $d = 2$, [18] for $d = 3$, [11] and [12] for $d \geqq 4$.

In this paper, we consider LERW on the pre-Sierpiński gasket and prove the following Theorems 1–3.

Theorem 1 *Loop-erased random walks on the finite pre-Sierpiński gaskets can be extended to a loop-erased random walk X on the infinite pre-Sierpiński gasket (the precise definition of X and a more precise statement are given in Section 5).*

Let $\lambda = (20 + \sqrt{205})/15$ and $\nu = \log 2/\log \lambda = 0.8375\ldots$.

Theorem 2 *For any $s > 0$, there exist positive constants $C_1(s)$ and $C_2(s)$ such that for any $n \in \mathbb{N}$,*

$$C_1(s) n^{s\nu} \leqq E[|X(n)|^s] \leqq C_2(s) n^{s\nu},$$

where $X(n)$ denotes the location of the LERW on the infinite pre-Sierpiński gasket starting at the origin after n steps and $|\cdot|$ the Euclidean distance.

ν is called the displacement exponent. We show also that the growth exponent is equal to $1/\nu$ (Proposition 15).

Theorem 3 *There are non-random positive constants C_3 and C_4 such that*

$$C_3 \leqq \overline{\lim_{n\to\infty}} \frac{|X(n)|}{\psi(n)} \leqq C_4, \ \ a.s.,$$

where $\psi(n) = n^\nu (\log\log n)^{1-\nu}$.

Our main tool for the proofs of the above results is the 'erasing-larger-loops-first' (ELLF) method, which was introduced to study the scaling limit, that is, the limit as the edge length tends to 0. The scaling limit for LERW on the Sierpiński gasket was obtained by two groups independently, by using different methods. For the 'standard' LERW on general graphs, the uniform spanning tree proves to be a powerful tool, which is used in [17]. By 'standard', we mean the loops are erased chronologically from a simple random walk as first introduced by G. Lawler. On the other hand, [3] constructed a LERW on the pre-Sierpiński gasket by ELLF, that is, by erasing loops in descending order of size of loops and proved that the resulting LERW has the same distribution as the 'standard' LERW. Furthermore, in [4], it is proved that ELLF works not only for simple random walks, but also for other kinds of random walks on some fractals, in particular, for self-repelling walks on the pre-Sierpiński gasket introduced in [2]. An important reason for this flexibility is that the ELLF method is based on self-similarity of the Sierpiński gasket.

Another advantage of the ELLF method is that it facilitates the extension of LERW to the infinite pre-Sierpiński gasket by providing a natural definition of two series of probability measures on sets of loopless paths. The extension is not trivial, for the simple random walk on the infinite pre-Sierpiński gasket is recurrent. The exact value of the displacement exponent has been deduced by a scaling argument ([1]). As for the proof of the existence, the authors erroneously wrote in [3] that Theorem 2 has been proved in [17], however, [17] deals with the scaling limit, not LERW on the infinite pre-Sierpiński gasket, and proves the short-time behavior of the limit process $\overline{X}(t)$:

Theorem 4 *(Theorem 7.10 in [17]) For any $p > 0$, there exist constants $C_5(p)$, $C_6(p) > 0$ such that for all $t \in [0, 1]$,*

$$C_5(p)\ t^{p\nu} \leqq E[|\overline{X}(t)|^p] \leqq C_6(p)\ t^{p\nu},$$

where $|\overline{X}(t)|$ denotes the Euclidean distance from the starting point at time t and $\nu = \log 2/\log \lambda$, $\lambda = (20 + \sqrt{205})/15$.

It is expected that the same exponent governs the long-time behavior of the walk, but no direct proof has been given. In order to know the displacement exponent, one has to look into how the scaled number of steps converges as the number of steps tends to infinity, not only the limit distribution. Thus, the author corrects her error and proves Theorem 2 in this paper by using the ELLF method.

The structure of the paper is as follows. In Section 2, we fix notation and in Section 3, we recall the ELLF method of loop-erasing. In Section 4 we establish some results on the asymptotics of the exit times from a series of triangles, which are used in later sections. In Section 5 we extend the walk to the infinite pre-Sierpiński gasket and prove Theorem 1. Finally, in Section 6 and Section 7, we prove Theorems 2 and 3, respectively.

2 Random walk on the pre-Sierpiński gaskets

2.1 The pre-Sierpiński gaskets

Let us recall the definition of the pre-Sierpiński gasket: let $O=(0,0)$, $a_0=(\frac{1}{2},\frac{\sqrt{3}}{2})$, $b_0=(1,0)$, $a_N=2^N a_0$ and $b_N=2^N b_0$ for $N\in\mathbb{N}$. Let F'_0 be the graph that consists of the three vertices and three edges of $\triangle O a_0 b_0$ and define a recursive sequence of graphs $\{F'_N\}_{N=0}^{\infty}$ by

$$F'_{N+1}=F'_N\cup(F'_N+a_N)\cup(F'_N+b_N),\quad N\in\mathbb{Z}_+=\{0,1,2,\ldots\},$$

where $A+a=\{x+a\ :\ x\in A\}$ and $kA=\{kx\ :\ x\in A\}$. F'_0, F'_1 and F'_2 are shown in Fig. 1.

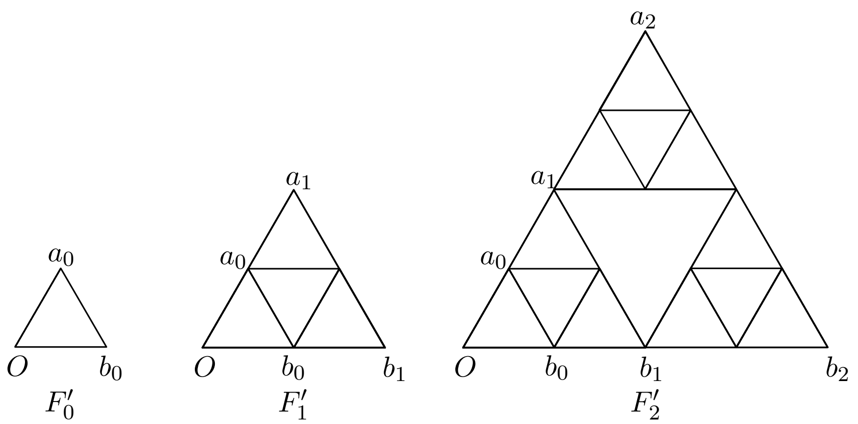

Fig 1: F'_0, F'_1 and F'_2.

Finally, we let F'^R_N denote the reflection of F'_N with respect to the y-axis, and $F_0=\bigcup_{N=1}^{\infty}(F'_N\cup F'^R_N)$; the graph F_0 is called the (infinite) **pre-Sierpiński gasket**. F_0 is shown in Fig. 2.

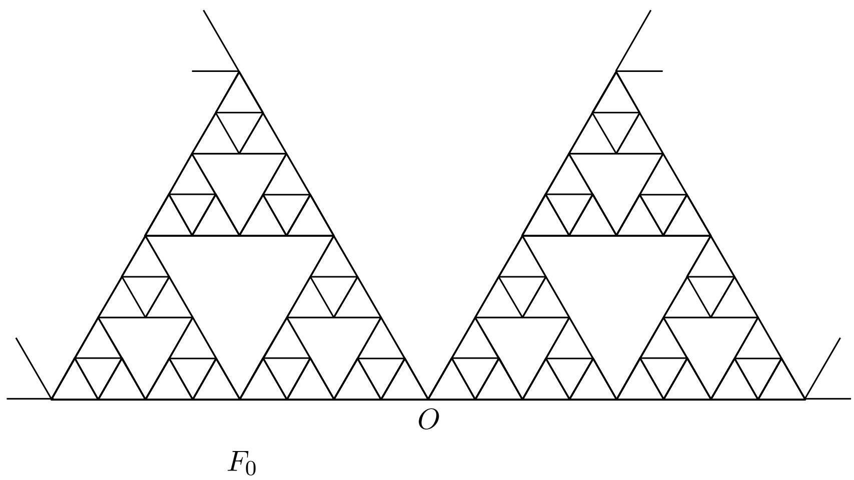

Fig 2: The pre-Sierpiński gasket F_0.

Furthermore, by letting G_0 and E_0 denote the set of vertices and the set of edges of F_0, respectively, we see that, for each $N\in\mathbb{Z}_+$, $F_N=2^N F_0$ can be regarded as a coarse graph with vertices $G_N=\{2^N x\ :\ x\in G_0\}$ and edges $E_N=\{2^N(x,y)\ :\ (x,y)\in E_0\}$. We call a (closed and filled) triangle which is a translation of $\triangle O a_M b_M$ and whose vertices are in G_M a 2^M–**triangle**.

2.2 Paths on the pre-Sierpiński gaskets

Define a set of finite paths on F_0 starting at O by

$$W = \{\ w = (w(0), w(1), \cdots, w(n)) :\ w(0) = O,\ (w(i-1), w(i)) \in E_0,\ 1 \leqq i \leqq n,\ n \in \mathbb{Z}_+\ \}.$$

This gives the natural definition for the length ℓ of a path $w = (w(0), w(1), \cdots, w(n)) \in W$; namely, $\ell(w) = n$.

For a path $w \in W$ and $A \subset G_0$, we define the hitting time of A by

$$T_A(w) = \inf\{j \geqq 0 :\ w(j) \in A\},$$

where we set $\inf \emptyset = \infty$. By taking $w \in W$ and $M \in \mathbb{Z}_+$, we will define a recursive sequence $\{T_i^M(w)\}_{i=0}^m$ of **hitting times of** G_M as follows: Let $T_0^M(w) = 0$, and for $i \geqq 1$, let

$$T_i^M(w) = \inf\{j > T_{i-1}^M(w) :\ w(j) \in G_M \setminus \{w(T_{i-1}^M(w))\}\};$$

here we take m to be the smallest integer such that $T_{m+1}^M(w) = \infty$. Then $T_i^M(w)$ can be interpreted as being the time (steps) taken for the path w to hit vertices in G_M for the $(i+1)$-st time, under the condition that if w hits the same vertex in G_M more than once in a row, we count it only once.

Now, we consider the following two sequences of subsets of W: for each $N \in \mathbb{Z}_+$, let

$$W_N = \{w = (w(0), w(1), \cdots, w(n)) \in W :\ w(T_1^N(w)) = a_N,\ n = T_1^N(w)\}$$

be the set of paths from O to a_N that do not hit any other vertices in G_N on the way and let

$$V_N = \{w = (w(0), w(1), \cdots, w(n)) \in W :\ w(T_1^N(w)) = b_N,\ w(T_2^N(w)) = a_N,\ n = T_2^N(w)\}$$

be the set of paths from O to a_N that hit b_N 'once' on the way (subject to the counting rule explained above).

Then, for a path $w \in W$ and each $M \in \mathbb{Z}_+$, we define the **coarse-graining map** Q_M by

$$(Q_M w)(i) = w(T_i^M(w)), \quad \text{for } i = 0, 1, 2, \ldots, m,$$

where m is the smallest integer such that $T_{m+1}^M(w) = \infty$ as above. Thus,

$$Q_M w = (w(T_0^M(w)), w(T_1^M(w)), \ldots, w(T_m^M(w)))$$

is a path on a coarser graph F_M. For $w \in W_N \cup V_N$ and $M \leqq N$, the end point of the coarse-grained path is $w(T_m^M(w)) = a_N$, and if we write $(2^{-M} Q_M w)(i) = 2^{-M} w(T_i^M(w))$, then $2^{-M} Q_M w$ is a path in $W_{N-M} \cup V_{N-M}$ and $\ell(2^{-M} Q_M w) = m$. In the following, we often write $w(T_i^M)$ for $w(T_i^M(w))$.

Define a family of probability measures P_N on W_N, $N \in \mathbb{N}$, by assigning each $w \in W_N$,

$$P_N[w] = \left(\frac{1}{4}\right)^{\ell(w)-1}.$$

(W_N, P_N) defines a family of fixed-end random walks Z_N on F_0 such that

$$Z_N(w)(i) = w(i), \quad i = 0, \cdots, \ell(w), \quad w \in W_N. \tag{2.1}$$

This is a simple random walk on F_0 starting at O and stopped at the first hitting time of a_N conditioned on the event that the walk does not hit any vertices in $G_N \setminus \{O\}$ on the way. The factor $(1/4)^{-1}$ comes from this conditioning.

Define another family of probability measures P'_N on V_N, $N \in \mathbb{N}$, by assigning each $w \in V_N$,

$$P'_N[w] = \left(\frac{1}{4}\right)^{\ell(w)-2}.$$

(V_N, P'_N) defines a family of fixed-end random walks Z'_N on F_0 such that

$$Z'_N(w)(i) = w(i), \quad i = 0, \cdots, \ell(w), \quad w \in V_N. \tag{2.2}$$

This is a simple random walk on F_0 started at O and stopped at the first hitting time of a_N conditioned on the event that the walk hits b_N 'once' on the way.

Note that a coarse grained simple random walk is again a simple random walk on a coarse graph, namely, if $M < N$, then the distributions of $2^{-M} Q_M Z_N$ and $2^{-M} Q_M Z'_N$ are equal to P_{N-M} and P'_{N-M}, respectively.

3 Loop erasure by the erasing-larger-loops-first rule

For $(w(0), w(1), \cdots, w(n)) \in W_N \cup V_N$, if there are $c \in G_0$, i and j, $0 \leqq i < j \leqq n$ such that $w(i) = w(j) = c$ and $w(k) \neq c$ for any $i < k < j$, we call the path segment $[w(i), w(i+1), \ldots, w(j)]$ a **loop formed at** c and define its **diameter** by $d = \max_{i \leqq k_1 < k_2 \leqq j} |w(k_1) - w(k_2)|$, where $|\ \cdot\ |$ denotes the Euclidean distance. Note that a loop can be a part of another larger loop formed at some other vertex. Suppose that for a loop $[w(i), w(i+1), \cdots, w(i+i_0)]$ there exists an $M' \in \mathbb{Z}_+$ such that

$$w(i) = w(i+i_0) \in G_{M'},\ d \geqq 2^{M'},$$

where d is the diameter of the loop. Let M be the maximum of such M' and call the loop a 2^M**-scale loop**. By definition, the paths in $W_N \cup V_N$ do not have any 2^N-scale loops. For each $N \in \mathbb{Z}_+$, let Γ_N be the set of loopless paths from O to a_N:

$$\Gamma_N = \{\ (w(0), w(1), \cdots, w(n)) \in W_N \cup V_N :\ w(i) \neq w(j),\ 0 \leqq i < j \leqq n,\ n \in \mathbb{N}\ \}.$$

Note that any loopless path in Γ_N is confined in $\triangle O a_N b_N$.

We now describe the loop-erasing procedure in a more organized manner than [3]. We start by erasing loops from paths in $W_1 \cup V_1$.

Loop erasure for $W_1 \cup V_1$

(i) Erase all the loops formed at O;

(ii) Progress one step forward along the path, and erase all the loops at the new position;

(iii) Iterate this process, taking another step forward along the path and erasing the loops there, until reaching a_1.

Let Lw denote the resulting path, where $L : W_1 \cup V_1 \to \Gamma_1$ is the loop-erasing operator. Fig. 3 shows all the possible loopless paths from O to a_1 on F_0. Here only the parts in $\triangle O a_1 b_1$ are shown, for it is impossible for any path to go into other triangles without making a loop. Note that $w \in W_1$ implies $Lw \in W_1 \cap \Gamma_1$, but that $w \in V_1$ can result in $Lw \in W_1 \cap \Gamma_1$, with b_1 being erased together with a loop. So far, our loop-erasing procedure is the same as the chronological method defined for paths on $\mathbb{Z}^d$ in [10].

For a general N, we erase loops from the largest-scale loops down, repeatedly applying the loop-erasing procedure for $W_1 \cup V_1$. To describe the procedure, we introduce a 'step-based' decomposition of a path based on the self-similarity and the symmetries of the pre-Sierpiński gaskets. Assume $w \in W_N \cup V_N$ and $0 \leqq M < N$. Note that the pair of adjacent 2^M–triangles

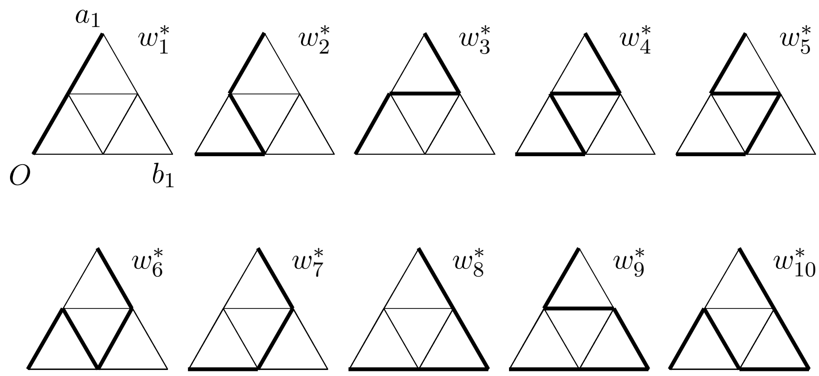

Fig 3: The loopless paths from O to a_1 on F_0.

including $(Q_M w)(i-1)$, $(Q_M w)(i)$ and $(Q_M w)(i+1)$ is similar to $F_0 \cap (\triangle O a_M b_M \cup \triangle O a_M^R b_M^R)$, where $\triangle O a_M^R b_M^R$ is the reflection of $\triangle O a_M b_M$ with regard to the y–axis. This leads to a unique decomposition:

$$(\tilde{w}; w_1, \cdots, w_{\ell(\tilde{w})}),\ \tilde{w} \in W_{N-M} \cup V_{N-M},\ w_i \in W_M,\ i = 1, \cdots, \ell(\tilde{w}) \tag{3.1}$$

such that $\tilde{w} = 2^{-M} Q_M w$ and that the path segment $(w(T^M_{i-1}(w)), w(T^M_{i-1}(w)+1), \cdots, w(T^M_i(w)))$ of w is identified with $w_i \in W_M$ by appropriate rotation, translation and reflection so that $w(T^M_{i-1}(w))$ is identified with O and $w(T^M_i(w))$ with a_M. Note that we may also need reflection acting only one of the triangle, to be more specific, after identifying the pair of 2^M–triangles with $\triangle O a_M b_M \cup \triangle O a_M^R b_M^R$, we may need to reflect the part of the path in $\triangle O a_M b_M$ with regard to $y = x/\sqrt{3}$ to obtain a path in $W_M \cup V_M$. We will use this kind of identification throughout the paper. We illustrate a simple example of the decomposition for $N = 2$ and $M = 1$ in Fig. 4.

Erasure of the largest-scale loops

(1) Decompose a path $w \in W_N \cup V_N$ as $(\tilde{w}; w_1, \cdots, w_{\ell(\tilde{w})})$, $\tilde{w} = 2^{-(N-1)} Q_{N-1} w \in W_1 \cup V_1$, $w_i \in W_{N-1}$, $i = 1, \cdots, \ell(\tilde{w})$ as in (3.1) with $M = N-1$. Fig. 5(a) shows the original w and Fig. 5(b) shows $Q_{N-1} w$.

(2) Erase all the loops from $\tilde{w}$ by following the loop-erasure for $W_1 \cup V_1$ to obtain $L\tilde{w} \in \Gamma_1$. Denote the coarse, loopless path $2^{(N-1)} L\tilde{w}$ on F_{N-1} by $\hat{Q}_{N-1} w$ (Fig. 5(c)). To be more precise, $\hat{Q}_{N-1} w$ can be expressed as

$$\hat{Q}_{N-1} w = (w(T^{N-1}_0), w(T^{N-1}_{s_1}), \cdots, w(T^{N-1}_{s_n})),$$

where $s_0 = 0$ and for $i \geqq 1$,

$$s_i = \sup\{\, j \ :\ w(T^{N-1}_j) = w(T^{N-1}_{s_{j-1}+1}) \,\}.$$

(3) Restore the original fine structures to the remaining parts as shown in Fig. 5(d) to obtain a path $w' \in W_N \cup V_N$. To be more precise, for each step i of $\hat{Q}_{N-1} w$, between $w(T^{N-1}_{s_i})$ and $w(T^{N-1}_{s_{i+1}})$, insert the path segment $w_{s_i+1} = (w(T^{N-1}_{s_i}), w(T^{N-1}_{s_i}+1), \cdots, w(T^{N-1}_{s_i+1}))$ chosen from the original decomposition in Step (1). Note that $Q_{N-1} w' = \hat{Q}_{N-1} w$ holds.

At this stage all the 2^{N-1}–scale loops have been erased. We repeat Procedure (1)–(3) within each 2^{N-1}–triangle to erase all the 2^{N-2}–scale loops, and then within each 2^{N-2}–triangle, and so on, until there remain no loops.

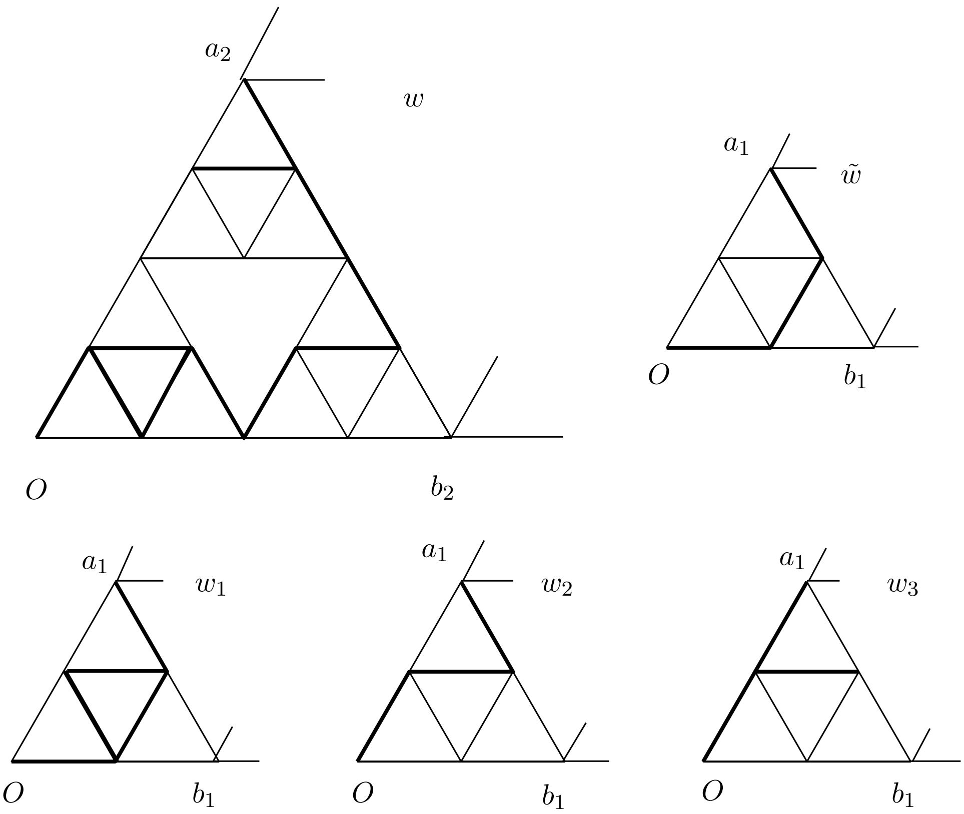

Fig 4: $w, \tilde{w}, w_1, w_2, w_3$.

To give a more precise description of the procedure, we prepare another kind of decomposition, a 'triangle-based' decomposition. For $w \in W_N$ and $0 \leqq M \leqq N$, we define the sequence $(\Delta_1, \dots, \Delta_k)$ of the 2^M–triangles w 'passes through', and their exit times $\{T_i^{ex,M}(w)\}_{i=1}^k$ as a subsequence of $\{T_i^M(w)\}_{i=1}^m$ as follows: Let $T_0^{ex,M}(w) = 0$. There is a unique 2^M–triangle that contains $w(T_0^M)$ and $w(T_1^M)$, which we denote by Δ_1. For $i \geqq 1$, define

$$J(i) = \min\{j \geqq 0 \ : \ j < m,\ T_j^M(w) > T_{i-1}^{ex,M}(w),\ w(T_{j+1}^M(w)) \notin \Delta_i\},$$

if the minimum exists, otherwise $J(i) = m$. Then define $T_i^{ex,M} = T_i^{ex,M}(w) = T_{J(i)}^M(w)$, and if $J(i) < m$ let Δ_{i+1} be the unique 2^M–triangle that contains both $w(T_i^{ex,M})$ and $w(T_{J(i)+1}^M)$. By definition, we see that $\Delta_i \cap \Delta_{i+1}$ is a one-point set $\{w(T_i^{ex,M})\}$, for $i = 1, \dots, k-1$. We denote the sequence of these triangles by $\sigma_M(w) = (\Delta_1, \dots, \Delta_k)$, and call it the 2^M–**skeleton** of w. We call the sequence $\{T_i^{ex,M}(w)\}_{i=0}^k$ **exit times** from the triangles in the skeleton. For each i, there is an $n = n(i)$ such that $T_{i-1}^{ex,M}(w) = T_n^M(w)$. If $T_i^{ex,M}(w) = T_{n+1}^M(w)$, we say that $\Delta_i \in \sigma_M(w)$ is **Type 1**, and if $T_i^{ex,M}(w) = T_{n+2}^M(w)$, **Type 2**. For $w \in W_N \cup V_N$ and $M < N$, if $Q_M w$ is similar to a path in Γ_{N-M}, namely, $2^{-M} Q_M w \in \Gamma_{N-M}$, then its 2^M–skeleton is a collection of distinct 2^M–triangles and each of them is either Type 1 or Type 2.

Assume $w \in W_N \cup V_N$ and $M \leqq N$. For each Δ_i in $\sigma_M(w)$, the **path segment of** w **in** Δ_i is defined by

$$w|_{\Delta_i} = [w(n),\ T_{i-1}^{ex,M}(w) \leqq n \leqq T_i^{ex,M}(w)]. \tag{3.2}$$

Note that the definition of $T_i^{ex,M}(w)$ allows a path segment $w|_{\Delta_i}$ to leak into the neighboring 2^M–triangles. If $Q_M w$ is similar to a path in Γ_{N-M}, then $w|_{\Delta_i} \in W_M$ or $w|_{\Delta_i} \in V_M$ (identification implied), according to the type of Δ_i, where the entrance to Δ_i is identified with O and the

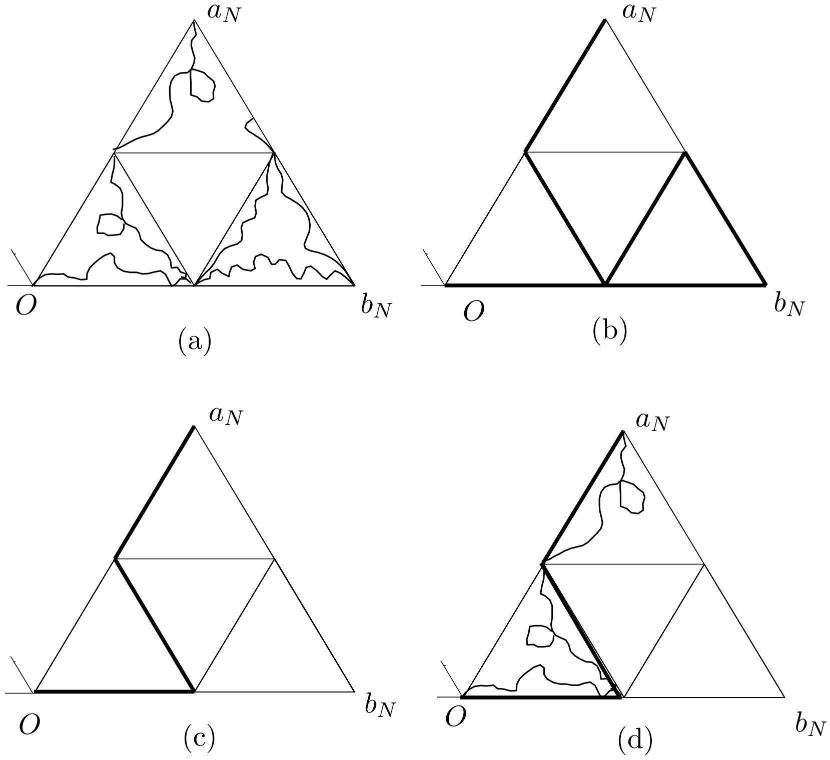

Fig 5: The loop-erasing procedure: (a) w, (b) $Q_{N-1}w$, (c) $\hat{Q}_{N-1}w$, (d) fine structures restored.

exit with a_M. This means that each w such that $Q_M w$ is similar to a path in Γ_{N-M} can be decomposed uniquely to

$$(\sigma_M(w);\ w|_{\Delta_1}, \cdots, w|_{\Delta_k}), \quad w|_{\Delta_i} \in W_M \cup V_M,\ i = 1, \cdots, k. \tag{3.3}$$

Induction step of loop erasure

Let $w \in W_N \cup V_N$ and $1 \leqq M < N$. Suppose that all of the 2^{N-1} to 2^{N-M}–scale loops have been erased from w, and denote the path obtained at this stage by $w' \in W_N \cup V_N$. Note that $Q_{N-M}w'$ is similar to a path in Γ_M.

1) Decompose w' to obtain $(\sigma_{N-M}(w');\ w'_1, \cdots w'_k)$, $w'_i \in W_{N-M} \cup V_{N-M}$ as given in (3.3).

2) From each w'_i, erase 2^{N-M-1}–scale loops (largest-scale loops) according to the base step procedure (1)–(3) above to obtain $\tilde{w}'_i \in W_{N-M} \cup V_{N-M}$.

3) Assemble $(\sigma_{N-M}(w');\ \tilde{w}'_1, \cdots, \tilde{w}'_k)$ to obtain $w'' \in W_N \cup V_N$, which is determined uniquely. w'' has no 2^{N-1} to 2^{N-M-1}–scale loops.

□

We repeat 1)–3) until we have no loops, and let $Lw \in \Gamma_N$ denote the resulting loopless path. In this way, the loop erasing operator L, first defined for $W_1 \cup V_1$, has been extended to $L : \bigcup_{N=1}^{\infty}(W_N \cup V_N) \to \bigcup_{N=1}^{\infty} \Gamma_N$ with $L(W_N \cup V_N) = \Gamma_N$. Note that the operation described above is essentially a repetition of loop-erasing for $W_1 \cup V_1$.

The operator L induces measures $\hat{P}_N = P_N \circ L^{-1}$ and $\hat{P}'_N = P'_N \circ L^{-1}$, which satisfy $\hat{P}_N[\Gamma_N] = \hat{P}'_N[\Gamma_N] = 1$. For $w_1^*, \cdots, w_{10}^*$ shown in Fig. 3, let

$$p_i = \hat{P}_1[w_i^*] = P_1[w : Lw = w_i^*], \quad q_i = \hat{P}'_1[w_i^*] = P'_1[w : Lw = w_i^*].$$

A direct calculation gives ([3]):

$$p_1 = 1/2,\ \ p_2 = p_3 = p_7 = 2/15,\ \ p_4 = p_5 = p_6 = 1/30,\ \ p_8 = p_9 = p_{10} = 0, \tag{3.4}$$

$$q_1 = 1/9,\ \ q_2 = q_3 = 11/90,\ \ q_4 = q_5 = q_6 = 2/45,\ \ q_7 = 8/45,\ \ q_8 = 2/9,\ \ q_9 = q_{10} = 1/18. \tag{3.5}$$

$\hat{P}_N$ and $\hat{P}'_N$ define two kinds of walks $Y_N = LZ_N$ and $Y'_N = LZ'_N$ on $F_0 \cap \triangle Oa_Nb_N$ obtained by erasing loops from the simple random walks Z_N and Z'_N, respectively.

For $w \in W_N \cup V_N$, we defined $\hat{Q}_{N-1}w$ in Step (2) for the erasure of the largest-scale loops. For later use we define $\hat{Q}_{N-K}w$ on F_{N-K} for all $K = 0, 1, \cdots, N$. Repeat the induction step 1)–3) K times to have down to 2^{N-K}–scale loops erased and denote the resulting path w'. Let $\hat{Q}_{N-K}w = Q_{N-K}w'$, namely, the coarse path before restoring fine structures. In particular, $\hat{Q}_N w = Q_N w$ and $\hat{Q}_0 w = Lw$. By construction, the distributions of $2^{-(N-K)}\hat{Q}_{N-K}Z_N$ and $2^{-(N-K)}\hat{Q}_{N-K}Z'_N$ equal $\hat{P}_K$ and $\hat{P}'_K$, respectively.

To compare the LERW defined here and the 'standard' LERW studied in [17], let us consider a simple random walk on the finite graph $F'_N = F_0 \cap \triangle Oa_Nb_N$, starting at O and stopped at the first hitting time of a_N. Let us denote this random walk by $\tilde{X}_N$. $\tilde{X}_N$ may visit O and b_N as many times as it likes, thus it may have 2^N–scale loops. If we coarse-grain the walk to obtain $Q_N\tilde{X}_N$ and erase 2^N–scale loops chronologically from it, we obtain a loopless walk $\hat{Q}_N\tilde{X}_N$, which is either (O, a_N) or (O, b_N, a_N). By direct calculation we have

$$\tilde{P}[\hat{Q}_N\tilde{X}_N = (O, a_N)] = \frac{2}{3},\ \tilde{P}[\hat{Q}_N\tilde{X}_N = (O, b_N, a_N)] = \frac{1}{3},$$

where $\tilde{P}$ denotes the law of the simple random walk defined above. Random walks Z_N and Z'_N considered in this paper can leak into neighboring 2^N–triangles, which parts are considered to be folded back into $\triangle Oa_Nb_N$ for $\tilde{X}_N$. Thus $\tilde{P}[\ \cdot\ \mid \hat{Q}_N\tilde{X}_N = (O, a_N)] = P_N$ and $\tilde{P}[\ \cdot\ \mid \hat{Q}_N\tilde{X}_N = (O, b_N, a_N)] = P'_N$, and therefore

$$\tilde{P} \circ L^{-1} = \frac{2}{3}\hat{P}_N + \frac{1}{3}\hat{P}'_N,$$

which equals the law of the 'standard' LERW.

4 Asymptotic behavior of the exit times

In this section, we look into the asymptotics for the exit times $T_1^{ex,N}(Y_N)$ and $T_1^{ex,N}(Y'_N)$ as $N \to \infty$, which will be used in Section 6.

For $w \in \Gamma_N$, let $s_1(w)$ and $s_2(w)$ denote the number of 2^0– triangles of Type 1 (the path passes two of the vertices) and those of Type 2 (the path passes all three vertices) in $\sigma_0(w)$, respectively. Note that $T_1^{ex,N}(w) = \ell(w) = s_1(w) + 2s_2(w)$. Define two sequences, $\{\Phi_N^{(1)}\}_{N\in\mathbb{N}}$ and $\{\Phi_N^{(2)}\}_{N\in\mathbb{N}}$, of generating functions by:

$$\Phi_N^{(1)}(x, y) = \sum_{w\in\Gamma_N} \hat{P}_N(w)x^{s_1(w)}y^{s_2(w)},$$

$$\Phi_N^{(2)}(x, y) = \sum_{w\in\Gamma_N} \hat{P}'_N(w)x^{s_1(w)}y^{s_2(w)}, \quad x, y \in \mathbb{C}.$$

For simplicity, we will write $\Phi^{(1)}(x, y)$ and $\Phi^{(2)}(x, y)$ for $\Phi_1^{(1)}(x, y)$ and $\Phi_1^{(2)}(x, y)$. A crucial observation is that in the process of erasing loops from Z_{N+1}, if we stop at the stage where we have obtained $\hat{Q}_1 Z_{N+1}$ after erasing down to 2^1–scale loops, it is nothing but the procedure for obtaining LZ_N from Z_N, in other words, the distribution of $2^{-1}\hat{Q}_1 Z_{N+1}$ equals $\hat{P}_N$. The same holds for $2^{-1}\hat{Q}_1 Z'_{N+1}$ and $\hat{P}'_N$ as well. This fact combined with (3.4) and (3.5) leads to the recursion relations for the generating functions given below:

Proposition 5 *(Proposition 3 in [3])*
The above generating functions satisfy the following recursion relations for all $N \in \mathbb{N}$ *:*

$$\Phi^{(1)}(x,y) = \frac{1}{30}(15x^2 + 8xy + y^2 + 2x^2y + 4x^3),$$

$$\Phi^{(2)}(x,y) = \frac{1}{45}(5x^2 + 11xy + 2y^2 + 14x^2y + 8x^3 + 5xy^2);$$

$$\Phi^{(i)}_{N+1}(x,y) = \Phi^{(i)}_N(\Phi^{(1)}(x,y), \Phi^{(2)}(x,y)), \quad i = 1,2.$$

In particular, inductively it also holds that for any $N, M \in \mathbb{N}$,

$$\Phi^{(i)}_{N+M}(x,y) = \Phi^{(i)}_N(\Phi^{(1)}_M(x,y), \Phi^{(2)}_M(x,y)), \quad i = 1,2.$$

Define the mean matrix by

$$\mathbf{M} = \begin{bmatrix} \frac{\partial}{\partial x}\Phi^{(1)}(1,1) & \frac{\partial}{\partial y}\Phi^{(1)}(1,1) \\ \frac{\partial}{\partial x}\Phi^{(2)}(1,1) & \frac{\partial}{\partial y}\Phi^{(2)}(1,1) \end{bmatrix} = \begin{bmatrix} \frac{9}{5} & \frac{2}{5} \\ \frac{26}{15} & \frac{13}{15} \end{bmatrix}. \tag{4.1}$$

It is strictly positive, and the larger eigenvalue is given by $\lambda = (20 + \sqrt{205})/15 = 2.2878\ldots$. The following is a restatement of Proposition 9 in [3].

Proposition 6 *(1) Let* $G^{(1)}_N(t)$ *and* $G^{(2)}_N(t)$ *be the Laplace transforms of* $\lambda^{-N}T_1^{ex,N}(Y_N)$ *and* $\lambda^{-N}T_1^{ex,N}(Y'_N)$*, respectively, that is,*

$$G^{(1)}_N(t) = \hat{E}_N[\exp(-t\lambda^{-N}T_1^{ex,N})],$$

$$G^{(2)}_N(t) = \hat{E}'_N[\exp(-t\lambda^{-N}T_1^{ex,N})], \quad t \in \mathbb{C}$$

where $\hat{E}_N$ *and* $\hat{E}'_N$ *are expectations with regard to* $\hat{P}_N$ *and* $\hat{P}'_N$*, respectively. Then they are expressed in terms of the generating functions as*

$$G^{(i)}_N(t) = \Phi^{(i)}_N(e^{-\lambda^{-N}t},\ e^{-2\lambda^{-N}t}) \quad i = 1,2. \tag{4.2}$$

(2) For each i, $G^{(i)}_N(t)$ *converges to an entire function* $g_i(t)$ *uniformly on any compact set in* $\mathbb{C}$ *as* $N \to \infty$. $g_1(t)$ *and* $g_2(t)$ *are the unique solution to*

$$g_1(\lambda t) = \Phi^{(1)}(g_1(t), g_2(t)), \quad g_2(\lambda t) = \Phi^{(2)}(g_1(t), g_2(t)), \ g_1(0) = g_2(0) = 1.$$

(3) $\lambda^{-N}T_1^{ex,N}(Y_N)$ *and* $\lambda^{-N}T_1^{ex,N}(Y'_N)$ *converge in law to some integrable random variables* T_1^* *and* T_2^**, whose Laplace transforms are given by* g_1 *and* g_2*, respectively, as* $N \to \infty$. T_1^* *and* T_2^* *have strictly positive probability density functions on* $(0,\infty)$. $\hat{E}_N[\ \lambda^{-N}T_1^{ex,N}\ ]$ *and* $\hat{E}'_N[\ \lambda^{-N}T_1^{ex,N}\ ]$ *converge to the expectations of* T_1^* *and* T_2^* *as* $N \to \infty$*, respectively.*

Next we want to obtain the left tail behavior of the scaled exit times and begin by estimating the Laplace transforms.

Proposition 7 *There exist positive constants* $C_{4.1}$, $C_{4.2}$ *and* t_0 *such that*

$$\exp(-C_{4.2}s^{\nu}) \leqq G^{(i)}_N(s) \leqq \exp(-C_{4.1}s^{\nu}), \ i = 1,2, \tag{4.3}$$

for all $t > t_0$ *and* $N > \log_\lambda(t/t_0)$.

Proof. Using (4.2), we rewrite the recursion in Proposition 5 as

$$G^{(i)}_{N+M}(t) = \Phi^{(i)}_M(G^{(1)}_N(t/\lambda^M), G^{(2)}_N(t/\lambda^M)), \quad i = 1, 2. \tag{4.4}$$

From the explicit forms of $\Phi^{(i)}$, $i = 1, 2$, in Proposition 5, it follows that for $0 < x, y < 1$,

$$q_1(x \wedge y)^2 \leqq \Phi^{(i)}(x, y) \leqq (x \vee y)^2, \quad i = 1, 2,$$

where $q_1 = 1/9$. Repeating this M times, we have

$$\{q_1(x \wedge y)\}^{2^M} \leqq \Phi^{(i)}_M(x, y) \leqq (x \vee y)^{2^M}, \quad i = 1, 2. \tag{4.5}$$

This combined with (4.4) gives

$$\{q_1(G^{(1)}_N(t/\lambda^M) \wedge G^{(2)}_N(t/\lambda^M))\}^{2^M} \leqq G^{(i)}_{N+M}(t) \leqq \{G^{(1)}_N(t/\lambda^M) \vee G^{(2)}_N(t/\lambda^M)\}^{2^M}. \tag{4.6}$$

Fix $t_0 > 0$ arbitrarily. Since $\{G^{(1)}_N(t_0) \vee G^{(2)}_N(t_0)\}^\infty_{N=1}$ and $\{(G^{(1)}_N(\lambda t_0) \wedge G^{(2)}_N(\lambda t_0)\}^\infty_{N=1}$ are strictly positive sequences and have strictly positive limits by Proposition 6 (2), there exist constants $c_1, c_2 \in (0, 1)$ such that

$$q_1(G^{(1)}_N(\lambda t_0) \wedge G^{(2)}_N(\lambda t_0)) > c_1, \quad G^{(1)}_N(t_0) \vee G^{(2)}_N(t_0) < c_2, \tag{4.7}$$

for all $N \in \mathbb{N}$. For any $t > t_0$, choose $M \in \mathbb{Z}_+$ such that

$$\lambda^M \leqq \frac{t}{t_0} < \lambda^{M+1}. \tag{4.8}$$

Then, the monotonicity of $G^{(i)}_N$ combined with (4.6), (4.7) and (4.8) gives

$$c_1^{2^M} \leqq G^{(i)}_{N+M}(t) \leqq c_2^{2^M}, \quad i = 1, 2.$$

This further leads to

$$\exp(-C_{4.2} t^\nu) \leqq G^{(i)}_N(t) \leqq \exp(-C_{4.1} t^\nu), \quad i = 1, 2$$

for all $t > t_0$ and $N > \log_\lambda(t/t_0)$, where we set $C_{4.2} = -\dfrac{\log c_1}{t_0^\nu}$ and $C_{4.1} = -\dfrac{\log c_2}{2t_0^\nu}$.

□

The following theorem plays an essential role in the proof of a law of the iterated logarithm in Section 7.

Theorem 8 *(Theorem 5.9 (ii) in [5])*
If the Laplace transforms of the scaled numbers of steps satisfy (4.3), the following hold:

(1) There exist positive constants $C_{4.3}$ and $C_{4.4}$ such that for any positive sequence $\{\alpha_N\}^\infty_{N=1}$ satisfying $\lim_{N\to\infty} 2^{N(1-\nu)/\nu}\alpha_N = \infty$ and $\lim_{N\to\infty} \alpha_N = 0$, the following holds:

$$\begin{aligned} -C_{4.3} &\leqq \varliminf_{N\to\infty} \alpha_N^{\nu/(1-\nu)} \log \hat{P}^{(i)}_N[\,\lambda^{-N} T_1^{ex,N}(w) \leqq \alpha_N\,] \\ &\leqq \varlimsup_{N\to\infty} \alpha_N^{\nu/(1-\nu)} \log \hat{P}^{(i)}_N[\,\lambda^{-N} T_1^{ex,N}(w) \leqq \alpha_N\,] \leqq -C_{4.4}, \quad i = 1, 2. \end{aligned}$$

(2) There exist positive constants $C_{4.5}$ – $C_{4.7}$ such that for any $\xi > 0$ and $N \in \mathbb{N}$ satisfying $(2^{\frac{1}{\nu}-1})^N \xi \geqq C_{4.5}$,

$$\hat{P}_N^{(i)}[\,\lambda^{-N} T_1^{ex,N}(w) \leqq \xi\,] \leqq C_{4.6} e^{-C_{4.7}\xi^{-\nu/(1-\nu)}}, \quad i = 1, 2$$

holds.

Theorem 8 is proved by using an exponential tauberian theorem. Theorem 9 below is a refrasing of the tauberian theorem (Theorem 2.2) in [7] into a convenient form for our purpose.

Theorem 9 *(Theorem A.10 in [5])*
Let f_1 and f_2 be concave functions on $(0, \infty)$, such that the Legendre transforms

$$f_i^*(\xi) = \inf_{s>0}(s\xi - f_i(s)), \quad \xi > 0, \quad i = 1, 2,$$

are non-decreasing and $f_i^*(\xi) > -\infty$ for any $\xi \in (0, \infty)$.
If a family of Borel probability measures $\{P_u,\ u > 0\}$ on $[0, \infty)$ satisfies

$$\begin{aligned} f_1(s) &\leqq \varliminf_{u\to\infty} -\frac{1}{u} \log \int_0^\infty e^{-us\xi} P_u[\,d\xi\,] \\ &\leqq \varlimsup_{u\to\infty} -\frac{1}{u} \log \int_0^\infty e^{-us\xi} P_u[\,d\xi\,] \leqq f_2(s), \quad s > 0, \end{aligned} \tag{4.9}$$

then

$$f_1^*(\xi_*(s^*(\xi))) \leqq \varliminf_{u\to\infty} \frac{1}{u} \log P_u[[0, \xi]] \leqq \varlimsup_{u\to\infty} \frac{1}{u} \log P_u[[0, \xi]] \leqq f_1^*(\xi), \quad \xi > 0$$

holds, where

$$s^*(\xi) = \sup\{s > 0\ :\ s\xi - f_2(s) \leqq f_1^*(\xi)\}, \quad \xi > 0, \tag{4.10}$$

and

$$\xi_*(s) = \inf\{\xi > 0\ :\ s\xi - f_1^*(\xi) \leqq f_2(s)\}, \quad s > 0. \tag{4.11}$$

Proof of Theorem 8.
We derive (1) from Theorem 9 for $i = 1$. Let

$$\mu_N([0, x]) := \hat{P}_N^{(1)}[\,\lambda^{-N} T_1^{ex,N}(w) \leqq \alpha_N x\,].$$

where $\{\alpha_N\}$ satisfis the assumption in Theorem 8(1). Let $\beta_N = {\alpha_N}^{-\nu/(1-\nu)}$, then by assumption $\{\beta_N\}$ is a divergent positive sequence. Since

$$\int_0^\infty e^{-\beta_N s\xi} \mu_N[\,d\xi\,] = \frac{1}{\alpha_N} G_N^{(1)}\left(\frac{\beta_N s}{\alpha_N}\right),$$

by Proposition 7 we have

$$\frac{1}{\alpha_N} \exp\left(-C_{4.2}\left(\frac{\beta_N s}{\alpha_N}\right)^\nu\right) \leqq \int_0^\infty e^{-\beta_N s\xi} \mu_N[\,d\xi\,] \leqq \frac{1}{\alpha_N} \exp\left(-C_{4.1}\left(\frac{\beta_N s}{\alpha_N}\right)^\nu\right),$$

for N large enough. Thus

$$-C_{4.2} s^\nu \leqq \varliminf_{N\to\infty} \frac{1}{\beta_N} \log \int_0^\infty e^{-\beta_N s\xi} \mu_N[\,d\xi\,] \leqq \varlimsup_{N\to\infty} \frac{1}{\beta_N} \log \int_0^\infty e^{-\beta_N s\xi} \mu_N[\,d\xi\,] \leqq -C_{4.1} s^\nu.$$

Then the assumptions for Theorem 9 hold with $u = \beta_N$, $P_u = \mu_N$, $f_2(s) = C_{4.2}s^\nu$ and $f_1(s) = C_{4.1}s^\nu$. In particular,

$$f_1^*(x) = -(C_{4.1}(1-\nu)^{1-\nu}\nu^\nu)^{1/(1-\nu)}x^{-\nu/(1-\nu)},$$

and

$$f_2^*(x) = -(C_{4.2}(1-\nu)^{1-\nu}\nu^\nu)^{1/(1-\nu)}x^{-\nu/(1-\nu)}.$$

Since $\xi_*(s^*(x)) = Bx$ with B being a positive constant depending only on $C_{4.1}$, $C_{4.2}$ and ν, Theorem 9 implies that

$$\begin{aligned} -C_{4.3}x^{-\nu/(1-\nu)} &\leqq \varliminf_{N\to\infty} \frac{1}{\beta_N} \log \mu_N([0,x]) \\ &\leqq \varlimsup_{N\to\infty} \frac{1}{\beta_N} \log \mu_N([0,x]) \leqq -C_{4.4}x^{-\nu/(1-\nu)}, \quad x > 0 \end{aligned}$$

holds for some positive constants $C_{4.3}$ and $C_{4.4}$. Setting $x = 1$, the conclusion of Theorem 8(1) follows. The same proof holds true for $i = 2$.

For (2), a combination of Proposition 7 and Chebyshev's inequality gives

$$\hat{P}_N^{(i)}[\,\lambda^{-N}T_1^{ex,N}(w) \leqq \xi\,] \leqq e^{-C_{4.1}s^\nu + \xi s},$$

for $s > t_0$ and $N > \log_\lambda(s/t_0)$. The right-hand side minimizes at $s_* = \left(\frac{C_{4.1}\nu}{\xi}\right)^{1/(1-\nu)}$, and $N > \log_\lambda(s_*/t_0)$ if $(2^{\frac{1}{\nu}-1})^N\xi \geqq \frac{C_{4.1}\nu}{t_0^{1-\nu}}$.

□

5 Extension to the infinite pre-Sierpiński gasket

In this section, we show that the loop-erased random walks defined in Section 3 can be extended to a loop-erased random walk on the infinite pre-Sierpiński gasket. For this purpose, we need walks from O to b_N as well as those from O to a_N. For each $N \in \mathbb{Z}_+$, let

$$W_N^b = \{w = (w(0), w(1), \cdots, w(n)) \in W :\ w(T_1^N(w)) = b_N,\ n = T_1^N(w)\},$$

$$V_N^b = \{w = (w(0), w(1), \cdots, w(n)) \in W :\ w(T_1^N(w)) = a_N,\ w(T_2^N(w)) = b_N,\ n = T_2^N(w)\}.$$

and probability measures $P_N^{(2)}$ on W_N^b and $P_N^{(4)}$ on V_N^b by

$$P_N^{(2)}[w] = \left(\frac{1}{4}\right)^{\ell(w)-1}, \quad w \in W_N^b,$$

$$P_N^{(4)}[w] = \left(\frac{1}{4}\right)^{\ell(w)-2}, \quad w \in V_N^b.$$

Let $U_N = W_N \cup V_N \cup W_N^b \cup V_N^b$ and extend the loop-erasing operator L so that it is defined on $\bigcup_{N=1}^\infty U_N$. Let $P_N^{(1)} = P_N$, $P_N^{(3)} = P'_N$ and $\hat{P}_N^{(i)} = P_N^{(i)} \circ L^{-1}$, for $i = 1,2,3,4$. In the rest of the paper, we will use the same notation Γ_N for loopless paths in U_N. Define a probability measure P_N^{rw} on U_N by

$$P_N^{rw} = \frac{11}{28}(P_N^{(1)} + P_N^{(2)}) + \frac{3}{28}(P_N^{(3)} + P_N^{(4)}), \tag{5.1}$$

and denote by $\tilde{Z}_N$ the conditioned simple random walk on F_0 defined by P_N^{rw} and let

$$\tilde{P}_N = P_N^{rw} \circ L^{-1}. \tag{5.2}$$

Then

$$\tilde{P}_N = \frac{11}{28}(\hat{P}_N^{(1)} + \hat{P}_N^{(2)}) + \frac{3}{28}(\hat{P}_N^{(3)} + \hat{P}_N^{(4)}), \tag{5.3}$$

and $\tilde{P}_N[\Gamma_N] = 1$.

Let

$$\Omega = \{ \ \omega = (\omega_0, \omega_1, \omega_2, \cdots) \ : \ \omega_0 \in \Gamma_0, \ \omega_N \in \Gamma_N, \ \omega_N|_{N-1} = \omega_{N-1}, \ N \in \mathbb{N} \ \},$$

where $\omega_N|_{N-1}$ denotes the path ω_N stopped at $T_1^{ex,N-1}(\omega_N)$. Note that $\Omega \in \bigotimes_{N\in\mathbb{Z}_+} 2^{\Gamma_N}$, which is the product σ–algebra on $\prod_{N\in\mathbb{Z}_+} \Gamma_N$. Let $\mathcal{B}$ be the σ-algebra on Ω generated by cylinder sets. Define the projection onto the first $N+1$ elements by

$$\pi_N \omega = (\omega_0, \omega_1, \ldots, \omega_N)$$

and a probability measure P_N^{prod} on $\pi_N\Omega$ by

$$P_N^{prod}[(\omega_0, \omega_1, \ldots, \omega_N)] = \tilde{P}_N[\omega_N]. \tag{5.4}$$

Proposition 10 *The sequence $\{P_N^{prod}\}$, $N \in \mathbb{Z}_+$ defined in (5.4) satisfies:*

$$P_N^{prod}[(\omega_0, \omega_1, \ldots, \omega_N)] = \sum_{\omega'} P_{N+1}^{prod}[(\omega_0, \omega_1, \ldots, \omega_N, \omega')], \tag{5.5}$$

where the sum is taken over all possible $\omega' \in \Gamma_{N+1}$ such that $\omega'|_N = \omega_N$.

Proof. Assume $u \in U_{N+1}$. Let $\Delta_0 = \triangle O a_0 b_0$ and $u_1 := (2^{-N}\hat{Q}_N u)|_{\Delta_0}$ be the path segment of $2^{-N}\hat{Q}_N u$ in Δ_0. Then $u_1 \in \Gamma_0 = \{(O,a_0),(O,b_0),(O,b_0,a_0),(O,a_0,b_0)\}$. Let $v_1^* = (O,a_0), v_2^* = (O,b_0), v_3^* = (O,b_0,a_0)$, $v_4^* = (O,a_0,b_0)$. Recall that in Step (2) of erasing the largest-scale loops from u, we obtain $\hat{Q}_N u$, which satisfies $2^{-N}\hat{Q}_N u \in \Gamma_1$ and whose law under $P_{N+1}^{(i)}$ is equal to $\hat{P}_1^{(i)}$. In particular, $P_{N+1}^{(i)}[\ u_1 = v_j^*\ ] = \hat{P}_1^{(i)}[\ v \in \Gamma_1 : v|_{\Delta_0} = v_j^*\ ]$. Let $\Delta = \triangle O a_N b_N$. For $\hat{w} \in \Gamma_N$, we classify the event $\{u \in U_{N+1} : Lu|_\Delta = \hat{w}\}$ by u_1. For $i = 1, 3$,

$$\begin{aligned}
\hat{P}_{N+1}^{(i)}[\ w \in \Gamma_{N+1} : w|_\Delta = \hat{w}\ ] &= P_{N+1}^{(i)}[\ u \in U_{N+1} : Lu|_\Delta = \hat{w}\ ] \\
&= \sum_{j=1}^{4} P_{N+1}^{(i)}[\ Lu|_\Delta = \hat{w} \mid u_1 = v_j^*\ ]\ P_{N+1}^{(i)}[\ u_1 = v_j^*\ ] \\
&= \sum_{j=1}^{4} \hat{P}_N^{(j)}[\hat{w}]\ \hat{P}_1^{(i)}[\ v \in \Gamma_1 : v|_{\Delta_0} = v_j^*\ ] \\
&= \hat{P}_N^{(1)}[\hat{w}]\ \hat{P}_1^{(i)}[\{w_1^*, w_3^*\}] + \hat{P}_N^{(2)}[\hat{w}]\ \hat{P}_1^{(i)}[\{w_5^*, w_7^*, w_8^*, w_9^*\}] \\
&\quad + \hat{P}_N^{(3)}[\hat{w}]\ \hat{P}_1^{(i)}[\{w_2^*, w_4^*\}] + \hat{P}_N^{(4)}[\hat{w}]\ \hat{P}_1^{(i)}[\{w_6^*, w_{10}^*\}],
\end{aligned}$$

where in the third equality we used the fact that under the condition that $u_1 = v_j^*$, the distribution of $Lu|_\Delta$ is equal to $\hat{P}_N^{(j)}$. Thus, by (3.4) and (3.5) we have

$$\hat{P}_{N+1}^{(1)}[\ w \in \Gamma_{N+1} : w|_\Delta = \hat{w}\ ] = \frac{19}{30}\hat{P}_N^{(1)}[\hat{w}] + \frac{1}{6}\hat{P}_N^{(2)}[\hat{w}] + \frac{1}{6}\hat{P}_N^{(3)}[\hat{w}] + \frac{1}{30}\hat{P}_N^{(4)}[\hat{w}],$$

$$\hat{P}_{N+1}^{(3)}[\ w \in \Gamma_{N+1} : w|_\Delta = \hat{w}\ ] = \frac{7}{30}\hat{P}_N^{(1)}[\hat{w}] + \frac{1}{2}\hat{P}_N^{(2)}[\hat{w}] + \frac{1}{6}\hat{P}_N^{(3)}[\hat{w}] + \frac{1}{10}\hat{P}_N^{(4)}[\hat{w}].$$

For $i = 2$, let $\hat{w}^R$ and v_i^{*R} be the paths obtained by reflection of $\hat{w}$ and v_i^* with regard to the line $y = x/\sqrt{3}$, respectively. Then using $v_1^{*R} = v_2^*$ and $v_3^{*R} = v_4^*$, we have

$$
\begin{aligned}
\hat{P}^{(2)}_{N+1}[\ w \in \Gamma_{N+1} : w|_\Delta = \hat{w}\ ] &= P^{(2)}_{N+1}[\ u \in U_{N+1} : Lu|_\Delta = \hat{w}\ ] \\
&= \sum_{j=1}^{4} P^{(2)}_{N+1}[\ Lu|_\Delta = \hat{w} \mid u_1 = v_j^*\ ]\ P^{(2)}_{N+1}[\ u_1 = v_j^*\ ] \\
&= \sum_{j=1}^{4} P^{(1)}_{N+1}[\ Lu|_\Delta = \hat{w}^R \mid u_1 = v_j^{*R}\ ]\ P^{(1)}_{N+1}[\ u_1 = v_j^{*R}\ ] \\
&= \hat{P}^{(2)}_N[\hat{w}^R]\ \hat{P}^{(1)}_1[\ v \in \Gamma_1 : v|_{\Delta_0} = v_2^*\ ] + \hat{P}^{(1)}_N[\hat{w}^R]\ \hat{P}^{(1)}_1[\ v \in \Gamma_1 : v|_{\Delta_0} = v_1^*\ ] \\
&\quad + \hat{P}^{(4)}_N[\hat{w}^R]\ \hat{P}^{(1)}_1[\ v \in \Gamma_1 : v|_{\Delta_0} = v_4^*\ ] + \hat{P}^{(3)}_N[\hat{w}^R]\ \hat{P}^{(1)}_1[\ v \in \Gamma_1 : v|_{\Delta_0} = v_3^*\ ] \\
&= \hat{P}^{(1)}_N[\hat{w}]\ \hat{P}^{(i)}_1[\{w_5^*, w_7^*, w_8^*, w_9^*\}] + \hat{P}^{(2)}_N[\hat{w}]\ \hat{P}^{(i)}_1[\{w_1^*, w_3^*\}] \\
&\quad + \hat{P}^{(3)}_N[\hat{w}]\ \hat{P}^{(i)}_1[\{w_6^*, w_{10}^*\}] + \hat{P}^{(4)}_N[\hat{w}]\ \hat{P}^{(i)}_1[\{w_2^*, w_4^*\}].
\end{aligned}
$$

Therefore,

$$
\hat{P}^{(2)}_{N+1}[\ w \in \Gamma_{N+1} : w|_\Delta = \hat{w}\ ] = \frac{1}{6}\hat{P}^{(1)}_N[\hat{w}] + \frac{19}{30}\hat{P}^{(2)}_N[\hat{w}] + \frac{1}{30}\hat{P}^{(3)}_N[\hat{w}] + \frac{1}{6}\hat{P}^{(4)}_N[\hat{w}],
$$

Similarly, we have

$$
\hat{P}^{(4)}_{N+1}[\ w \in \Gamma_{N+1} : w|_\Delta = \hat{w}\ ] = \frac{1}{2}\hat{P}^{(1)}_N[\hat{w}] + \frac{7}{30}\hat{P}^{(2)}_N[\hat{w}] + \frac{1}{10}\hat{P}^{(3)}_N[\hat{w}] + \frac{1}{6}\hat{P}^{(4)}_N[\hat{w}].
$$

We want a probability vector $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, i.e., $\alpha_i \geqq 0$, $i = 1, 2, 3, 4$ with $\sum_{i=1}^4 \alpha_i = 1$, that satisfies

$$
\sum_{i=1}^{4} \alpha_i \hat{P}^{(i)}_{N+1}[w|_\Delta = \hat{w}] = \sum_{i=1}^{4} \alpha_i \hat{P}^{(i)}_N[\hat{w}]
$$

for every $\hat{w} \in \Gamma_N$, $N \in \mathbb{Z}_+$. This equation can be rewritten in a simple manner as

$$
\alpha = \alpha \mathbf{P},
$$

where

$$
\mathbf{P} = \frac{1}{30}\begin{bmatrix} 19 & 5 & 5 & 1 \\ 5 & 19 & 1 & 5 \\ 7 & 15 & 5 & 3 \\ 15 & 7 & 3 & 5 \end{bmatrix}, \tag{5.6}
$$

and

$$
(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\frac{11}{28}, \frac{11}{28}, \frac{3}{28}, \frac{3}{28}\right)
$$

is the unique choice of the probability vector satisfying the above equation. □

Proposition 10 gives the consistency condition for Kolmogorov's extension theorem, and we have the existence of a unique probability measure P on $(\Omega, \mathcal{B})$, such that

$$
P \circ \pi_N^{-1} = P_N^{prod}. \tag{5.7}
$$

$(\Omega, \mathcal{B}, P)$ defines a loop-erased random walk X on F_0: For each $\omega = (\omega_0, \omega_1, \cdots) \in \Omega$ and $i \in \mathbb{Z}_+$, take N such that $i \leqq 2^N$ and let

$$
X(\omega)(i) = \omega_N(i).
$$

The right hand side is determined uniquely as long as $i \leqq 2^N$ holds, since $\omega_N|_{N-1} = \omega_{N-1}$ for any $N \in \mathbb{N}$. This completes the proof of Theorem 1.

To summarize, we established the following:

Theorem 1 *(restatement) On a measurable space $(\Omega, \mathcal{B})$, an infinite length LERW X on the infinite pre-Sierpiński gasket is constructed such that X stopped at the exit time of $\triangle Oa_Nb_N$ has the same law as $\tilde{P}_N$ for any $N \in \mathbb{Z}_+$.*

Remark 1

For $N \in \mathbb{N}$ and $w \in U_N$, let $u_N = 2^{-N}\hat{Q}_N w$ and let $u_M = (2^{-M}\hat{Q}_M w)|_{\Delta_0}$, $M = 0, 1, \cdots, N-1$, where $\hat{Q}_M$ is defined in the second last paragraph of Section 3 and $\Delta_0 = \triangle Oa_0b_0$. Note that $u_M \in \Gamma_0$. For any $M \leqq N-1$ and any $x_k \in \Gamma_0$, $k = M, M+1, \cdots, N$,

$$P_N^{(i)}[\ u_M = x_M \mid u_k = x_k,\ k = M+1, M+2, \cdots, N\ ] = P_N^{(i)}[\ u_M = x_M \mid u_{M+1} = x_{M+1}\ ]. \quad (5.8)$$

Thus, for each $i = 1, 2, 3, 4$, $P_N^{(i)}$, $N \in \mathbb{N}$ define a backward Markov chains on the state space $\Gamma_0 = \{v_1^*, v_2^*, v_3^*, v_4^*\}$ such that

$$P_N^{(i)}[\ u_N = v_i^*\ ] = 1,$$

and for $M \leqq N-1$,

$$P_N^{(i)}[\ u_M = v_j^* \mid u_{M+1} = v_k^*\ ] = P_{kj},$$

where P_{kj} denotes the (k, j)- element of the transition probability matrix, which coincides with $\mathbf{P}$ in (5.6).

$\alpha = \dfrac{1}{28}(11, 11, 3, 3)$ is the unique invariant probability vector, namely, the unique nonnegative solution to

$$\alpha = \alpha\mathbf{P}.$$

Furthermore, for any probability vector a, by the Perron-Frobenius theorem, it holds that

$$\lim_{n\to\infty} a\mathbf{P}^n = \alpha. \quad (5.9)$$

In terms of the loop-erased walk measures, the above fact can be expressed as

$$\hat{P}_{N+K}^{(i)}[\ w|_K \in A_K\ ] = \sum_{j=1}^{4} (\mathbf{P}^N)_{ij} \hat{P}_K^{(j)}[A_K],$$

where for $w \in \Gamma_{N+K}$, $w|_K$ denotes the path w stopped at $T_1^{ex,K}(w)$ and $A_K \subset \Gamma_K$. Thus, for any probability vector a, we have as $N \to \infty$,

$$\sum_{i=1}^{4} a_i \hat{P}_{N+K}^{(i)}[\ w|_K \in A_K\ ] \to \sum_{i=1}^{4} \alpha_i \hat{P}_K^{(i)}[A_K].$$

In particular, $a = \dfrac{1}{3}(0, 2, 0, 1)$ represents the 'standard' LERW studied in [17]. The 'standard' LERW and its scaling limit can be described intuitively as follows; we start with two paths $w_1 = (O, b_0)$ and $w_2 = (O, a_0, b_0)$ on $F_0 \cap \triangle Oa_0b_0$, with probability 2/3 and 1/3, respectively. Then paths w_1 and w_2 branch into smaller copies of paths $w_1^*, \cdots, w_{10}^*$ shown in Fig. 3 (reflection with regard to $y = x/\sqrt{3}$ implied), which are paths on $2^{-1}(F_0 \cap \triangle Oa_1b_1)$, according to $\hat{P}_1^{(2)}$ and $\hat{P}_1^{(4)}$, respectively. Inductively, for each LERW path occurring on the fine pre-Sierpiński gasket $2^{-N}(F_0 \cap \triangle Oa_Nb_N)$, by letting the path segment in each 2^{-N}–triangle (defined similarly to a 2^M–triangle, but with edge length 2^{-N}) branch into smaller copies of $w_1^*, \cdots, w_{10}^*$ (reflection and rotation implied), according to the type of the triangle ($\hat{P}_1^{(2)}$ for Type 1 and $\hat{P}_1^{(4)}$ for Type 2), we obtain a LERW on $2^{-(N+1)}(F_0 \cap \triangle Oa_{N+1}b_{N+1})$. In the limit as $N \to \infty$, the LERW converges to a continuous process on the Sierpiński gasket. To see the connection with (5.9), fix $K \in \mathbb{N}$ arbitrarily. Suppose we have repeated the branching procedure N times to obtain a LERW on $2^{-N}(F_0 \cap \triangle Oa_N b_N)$. (5.9) says that if we magnify the LERW with the factor of 2^N and look at the LERW stopped at the first exit time of $\triangle Oa_K b_K$, then the distribution approaches to that of $\tilde{Z}_K$ as $N \to \infty$.

6 Proof of Theorem 2

Let X be the loop-erased random walk defined in Section 5 and let

$$\tilde{\Phi}_N(x,y) = \frac{11}{14}\Phi_N^{(1)}(x,y) + \frac{3}{14}\Phi_N^{(2)}(x,y), \tag{6.1}$$

where $\Phi_N^{(i)}(x,y)$, $i = 1,2$ are defined in Section 4. The Laplace transform of $\lambda^{-N} T_1^{ex,N}(X)$ is given by

$$\tilde{g}_N(t) := \tilde{\Phi}_N(e^{-t\lambda^{-N}},\, e^{-2t\lambda^{-N}}) = \frac{11}{14}G_N^{(1)}(t) + \frac{3}{14}G_N^{(2)}(t). \tag{6.2}$$

Define for each $n \in \mathbb{N}$,

$$D_n(X) = \min\{M \geqq 0 :\ |X(i)| \leqq 2^M,\ 0 \leqq i \leqq n\},$$

and let $K = K(n)$ be the nonnegative integer such that

$$\lambda^K \leqq n < \lambda^{K+1} \tag{6.3}$$

holds.

Proposition 11 *(short-path estimate) There exist positive constants $C_{6.1}$ and $C_{6.2}$ such that*

$$P[\ D_n(X) < K(n) - M\ ] \leqq C_{6.1} e^{-C_{6.2}\lambda^M}$$

holds for any $n, M \in \mathbb{N}$ satisfying $K(n) > M$.

Proof. Take $C_{6.2} > 0$ arbitrarily. Since Proposition 6 (2) implies that $\{\tilde{g}_N(t)\}$ is a convergent sequence for any $t \in \mathbb{C}$, we can take $C_{6.1} > 0$ such that $\tilde{g}_N(-C_{6.2}) < C_{6.1}$ for all $N \in \mathbb{N}$. By Chebyshev's inequality, we have

$$\tilde{P}_N[\ \lambda^{-N} T_1^{ex,N}(X) \geqq \lambda^M\ ] \leqq \tilde{g}_N(-C_{6.2})\ e^{-C_{6.2}\lambda^M} < C_{6.1}\ e^{-C_{6.2}\lambda^M}.$$

This leads to

$$\begin{aligned} P[\ D_n(X) < K(n) - M\ ] &\leqq P[\ T_1^{ex,K-M}(X) > n\ ] \\ &= \tilde{P}_{K-M}[\ T_1^{ex,K-M}(w) > n\ ] \\ &\leqq \tilde{P}_{K-M}[\ \lambda^{-(K-M)} T_1^{ex,K-M}(w) > \lambda^M\ ] \\ &\leqq C_{6.1} e^{-C_{6.2}\lambda^M}. \end{aligned}$$

□

Proposition 12 *(long-path estimate) There exist positive constants $C_{6.3}$ and $C_{6.4}$ such that*

$$P[\ D_n(X) > K(n) + M\ ] \leqq C_{6.3} e^{-C_{6.4} 2^M}$$

for any $n, M \in \mathbb{N}$.

Proof. Since $P[\ D_n(X) > K(n)+M\ ] = 0$ for $n = 1, 2$, we may assume $n \geqq 3$ and hence $K(n) \in \mathbb{N}$. Note that

$$\begin{aligned} P[\ D_n(X) > K(n) + M\ ] &\leqq P[\ T_1^{ex,K+M}(X) < n\ ] \\ &= \tilde{P}_{K+M}[\ T_1^{ex,K+M}(w) < n\ ] \\ &\leqq \tilde{P}_{K+M}[\ T_1^{ex,K+M}(w) \leqq \lambda^{K+1}\ ]. \end{aligned}$$

Fix $0 < \delta < 1$ arbitrarily, then

$$\begin{aligned} \tilde{P}_{K+M}[\ T_1^{ex,K+M}(w) \leqq \lambda^{K+1}\ ] &= \sum_{w \in \Gamma_{K+M},\ \ell(w) \leqq \lambda^{K+1}} \tilde{P}_{K+M}[w] \\ &\leqq \delta^{-1} \sum_{w \in \Gamma_{K+M},\ \ell(w) \leqq \lambda^{K+1}} \tilde{P}_{K+M}[w]\ \delta^{\ell(w)\lambda^{-(K+1)}} \\ &\leqq \delta^{-1} \tilde{\Phi}_{K+M}(\delta^{\lambda^{-(K+1)}},\ \delta^{2\lambda^{-(K+1)}}), \end{aligned}$$

where we used $\ell(w) = s_1(w) + 2s_2(w)$ and the definition of the generating functions.

Let $t' = -\lambda^{-1} \log \delta > 0$. Since Proposition 6 (3) implies that $\tilde{\Phi}_N(\delta^{\lambda^{-(N+1)}},\ \delta^{2\lambda^{-(N+1)}}) = \tilde{g}_N(t')$ (is less than 1 and) converges as $N \to \infty$ to a limit strictly smaller than 1, we can choose $0 < r < 1$ such that

$$\Phi_N^{(i)}(\delta^{\lambda^{-(N+1)}},\ \delta^{2\lambda^{-(N+1)}}) < r, \quad i = 1, 2 \tag{6.4}$$

for all $N \in \mathbb{N}$. Thus,

$$\tilde{\Phi}_{K+M}(\delta^{\lambda^{-(K+1)}},\ \delta^{2\lambda^{-(K+1)}}) < \tilde{\Phi}_M(r, r) \leqq r^{2^M} = e^{-C_{6.4} 2^M},$$

where we used Proposition 5 and (4.5) in the last inequality and set $C_{6.4} = -\log r$. Taking $C_{6.3} = \delta^{-1}$ completes the proof.

□

To obtain the displacement exponent, we use the following inequality that holds for any $\mathbb{N}$–valued random variable Y and $s > 0$:

$$s\ C_{6.5}(s) \sum_{k=1}^{\infty} k^{s-1} P[\ Y \geqq k\ ] \leqq E[Y^s] \leqq s \sum_{k=1}^{\infty} k^{s-1} P[\ Y \geqq k\ ] + C_{6.6}(s), \tag{6.5}$$

where for $0 < s < 1$, $C_{6.5}(s) = 1$, $C_{6.6}(s) = 1$, for $s > 1$, $C_{6.5}(s) = \frac{1}{2^s}$, $C_{6.6}(s) = 0$, and $C_{6.5}(1) = 1$, $C_{6.6}(1) = 0$.

Let $\nu = \log 2 / \log \lambda$.

Proposition 13 *For any $s > 0$, there exists a positive constant $C_1(s)$ such that*

$$E[\ |X(n)|^s\ ] \geqq C_1(s)\ n^{s\nu}$$

for all $n \in \mathbb{N}$.

Proof. Fix $M_0 \in \mathbb{N}$ such that $C_{6.1} e^{-C_{6.2}\lambda^{M_0}} < 1/2$, where $C_{6.1}$ and $C_{6.2}$ are as in Proposition 11. Take n_1 large enough so that $K(n_1) > M_0 + 2$, where $K(n)$ is as in (6.3). Then for $n \geqq n_1$,

$$P[\ |X(n)| \leqq 2^{K-M_0-2}\ ] \leqq P[\ D_n < K - M_0\ ] < \frac{1}{2}. \tag{6.6}$$

Note that if we write $X(n) = (x_1, x_2)$, then $x_1 \in \frac{1}{2}\mathbb{Z}_+$ and $x_2 \in \frac{\sqrt{3}}{2}\mathbb{Z}_+$, thus $4|X(n)|^2 \in \mathbb{Z}_+$.

We give a proof in the case of $s > 2$. We make use of (6.5) with $Y = 4|X(n)|^2$.

$$\begin{aligned}
E[\ |X(n)|^s\ ] &= \frac{1}{2^s} E[\ (4|X(n)|^2)^{s/2}\ ] \\
&\geqq \frac{1}{2^s}\frac{s/2}{2^{s/2}} \sum_{k=1}^{\infty} k^{s/2-1} P[\ Y \geqq k\ ] \\
&\geqq \frac{s}{2^{3s/2+1}} \sum_{m=0}^{\infty} \sum_{k=4^{m+1}}^{4^{m+2}-1} (4^{m+1})^{s/2-1}\ P[\ Y > 4^{m+2}\ ] \\
&\geqq \frac{s}{2^{3s/2+1}} \sum_{m=0}^{\infty} (4^{m+1})^{s/2}\ P[\ Y > 4^{m+2}\ ] \\
&\geqq \frac{s}{2^{3s/2+1}} \sum_{m=0}^{\infty} (4^{m+1})^{s/2}\ P[\ |X(n)| > 2^{m+1}\ ] \\
&\geqq \frac{s}{2^{3s/2+1}}\ 2^{(K-M_0-2)s}\ P[\ |X(n)| > 2^{K-M_0-2}\ ] \\
&= \frac{s}{2^{3s/2+1}}\ 2^{(K-M_0-2)s}(1 - P[\ |X(n)| \leqq 2^{K-M_0-2}\ ]) \\
&\geqq s\ 2^{-(M_0 s+\frac{7}{2}s+2)}\ 2^{Ks} \geqq \tilde{C}_1(s) n^{s\nu}, \quad n \geqq n_1,
\end{aligned}$$

where we used (6.6) and set $\tilde{C}_1(s)$ is a positive constant that does not depend on n. Setting $C_1(s) = \tilde{C}_1(s) \wedge n_1^{-s\nu}$, we have the desired inequality for all $n \in \mathbb{N}$.

The case of $0 < s \leqq 2$ can be proved similarly.

□

Proposition 14 *For any $s > 0$, there exists a positive constant $C_2(s)$ such that*

$$E[\ |X(n)|^s\ ] \leqq C_2(s)\ n^{s\nu}$$

for all $n \in \mathbb{N}$.

Proof. First note that

$$P[\ |X(n)| \geqq 2^m\ ] \leqq P[\ D_n(X) > m-1\ ]. \tag{6.7}$$

In the case of $s > 2$, making use of (6.5) with $Y = 4|X(n)|^2$, we have

$$\begin{aligned}
E[\ |X(n)|^s\ ] &\leqq \frac{s}{2^{s+1}} \sum_{m=0}^{\infty} 4^{m+1}\ 4^{(m+2)(s/2-1)}\ P[\ Y \geqq 4^{m+1}\ ] \\
&= s2^{s-3} \sum_{m=0}^{\infty} 2^{sm}\ P[\ |X(n)| \geqq 2^m\ ] \\
&\leqq s2^{s-3}\Big(\sum_{m=0}^{K+1} 2^{sm}\ P[\ |X(n)| \geqq 2^m\ ] + \sum_{m=K+2}^{\infty} 2^{sm}\ P[\ |X(n)| \geqq 2^m\ ] \Big) \\
&\leqq s2^{s-3}\Big(\sum_{m=0}^{K+1} 2^{sm} + \sum_{m=K+2}^{\infty} 2^{sm}\ P[\ D_n(X) > m-1\ ] \Big) \qquad \text{(use of (6.7))} \\
&\leqq c_1(s)2^{Ks} + s2^{s-3}C_{6.3}\ 2^{s(K+1)} \sum_{\ell=1}^{\infty} 2^{\ell s} e^{-C_{6.4}2^{\ell}} \qquad \text{(by Proposition 12)} \\
&\leqq C_2(s)n^{s\nu},
\end{aligned}$$

where $c_1(s)$ and $C_2(s)$ are positive constants depending only on s and we used the convergence of the series above. The case for $0 < s \leqq 2$ can be proved similarly. □

Proposition 14 combined with Proposition 13 gives Theorem 2.

Remark 2
We can show that the growth exponent is equal to $1/\nu$. Define

$$M(m) := \min\{n \in \mathbb{N} \ : \ |X(n)| > m\},$$

then we have the following.

Proposition 15 *There exist positive constants C_7 and C_8 such that for all $m \in \mathbb{N}$*

$$C_7 \ m^{1/\nu} \leqq E[M(m)] \leqq C_8 \ m^{1/\nu}$$

holds.

Proof. Note that

$$E[M(2^N)] = E[\,T_1^{ex,N}(X)\,] + 1 = \tilde{E}_N[\,T_1^{ex,N}(X)\,] + 1,$$

where $\tilde{E}_N$ denotes expectation with regard to $\tilde{P}_N$. Proposition 6(3) implies that

$$\lim_{N\to\infty} \frac{E[\,M(2^N)\,]}{(2^N)^{1/\nu}}$$

exists and is positive and finite. Then the monotonicity of $E[M(m)]$ in m gives the statement. □

Remark 3
In [6], the 'standard' self-avoiding walk, which is defined by the uniform measure on self-avoiding paths of a given length, is studied. They showed the existence of the exponent in the form of

$$\lim_{n\to\infty} \frac{\log E_n[|X'(n)|^s]}{\log n} = s\,\nu_{\text{SAW}}, \quad s > 0 \tag{6.8}$$

where $|X'(n)|$ denotes the end-to-end distance of an n-step self-avoiding path, and $\nu_{\text{SAW}} = \log 2/\log(\frac{7-\sqrt{5}}{2})$. The fact that the exponent ν in Theorem 2 is different from ν_{SAW} shows that LERW is in a different universality class from the self-avoiding walk. Note also that self-avoiding walk cannot be extended to that of infinite length, for the consistency condition is not satisfied because of culs-de-sac, thus the expectation in (6.8) is taken over the uniform measure on the n-step self-avoiding paths.

7 Proof of Theorem 3

In this section, we prove the law of the iterated logarithm. First we prove the upper bound:

Proposition 16 *There exists a non-random positive constant C_4 such that*

$$\overline{\lim_{n\to\infty}} \frac{|X(n)|}{\psi(n)} \leqq C_4, \quad P\text{– } a.s.,$$

where $\psi(n) = n^\nu (\log\log n)^{1-\nu}$.

Proof. Let $\hat{\mu}_N$ be the distribution of $\lambda^{-N}T_1^{ex,N}(X)$ under P. For each $x > 1$ there is a unique integer N such that $2^N \leqq x < 2^{N+1}$. For $k > 0$ satisfying $2^{-N}k \geqq C_{4.5}$, Proposition 7 and Theorem 8 (2) imply that

$$\begin{aligned} P[\max_{0\leqq j\leqq k}|X(j)| > x\,] &\leqq P[\,T_1^{ex,N}(X) \leqq k\,] \\ &= \hat{\mu}_N([0, \lambda^{-N}k]]) \\ &\leqq C_{4.6}e^{-C_{4.7}(xk^{-\nu}/2)^{1/(1-\nu)}}. \end{aligned}$$

Let $\gamma > 1$ and $A > 0$. For $m \in \mathbb{N}$, let $x = A\psi(\gamma^m)$ and k be the largest integer that does not exceed γ^{m+1}. Then for all m large enough, the condition $2^{-N}k \geqq C_{4.5}$ is satisfied, thus we can apply the above inequality to have

$$P[\max_{0\leqq j\leqq \gamma^{m+1}}|X(j)| > A\psi(\gamma^m)\,] \leqq C_{4.6}e^{-C_{4.7}(xk^{-\nu}/2)^{1/(1-\nu)}} \leqq \frac{c_3}{m^\alpha},$$

where $\alpha = C_{4.7}\left(\dfrac{A}{2\gamma^\nu}\right)^{1/(1-\nu)}$ and c_3 is a positive constant independent of m. Thus,

$$\begin{aligned} \sum_{m=1}^{\infty} P[\max_{\gamma^m < j\leqq \gamma^{m+1}}|X(j)| > A\psi(\gamma^m)\,] &\leqq \sum_{m=1}^{\infty} P[\max_{0\leqq j\leqq \gamma^{m+1}}|X(j)| > A\psi(\gamma^m)\,] \\ &\leqq c_4 + c_3\sum_{m=1}^{\infty}\frac{1}{m^\alpha}, \end{aligned}$$

for some constant $c_4 > 0$. The sequence $\displaystyle\sum_{m=1}^{\infty}\frac{1}{m^\alpha}$ converges if we take A large enough so that $\alpha > 1$. The rest is a usual Borel-Cantelli argument and the statement holds with $C_4 = A$. □

Now we show the lower bound:

Proposition 17 *There exists a non-random positive constant C_3 such that*

$$C_3 \leqq \overline{\lim_{n\to\infty}}\frac{|X(n)|}{\psi(n)}, \quad P\text{- } a.s.$$

holds.

Note that we cannot use the Markov property, which is essential for establishing the lower bound for simple random walk on $\mathbb{Z}^d$. However, ELLF construction allows us to make use of a 'Markov-like structure' in the scale direction to obtain the result. Our proof follows closely the proof of the law of the iterated logarithm for stochastic chains on $\mathbb{Z}$ in [5], except that we do not have the independence of $T_1^{ex,M+1}(X) - T^{ex,M}(X)$, $M \in \mathbb{Z}_+$.

Lemma 18 *If there exists a positive constant c such that*

$$P[\bigcap_{N=1}^{\infty}\bigcup_{M=N}^{\infty}\{(\log M)^{(1-\nu)/\nu}\lambda^{-M}T_1^{ex,M}(X) \leqq c\}\,] = 1, \tag{7.1}$$

then it holds that

$$\overline{\lim_{n\to\infty}}\frac{|X(n)|}{\psi(n)} \geqq c^{-\nu}, \quad P\text{- } a.s..$$

Proof. The assumption implies that for P–almost all $\omega \in \Omega$, there exists an increasing sequence $M'_k = M'_k(\omega)$, $k = 1, 2, \dots$ such that

$$(\log M'_k)^{(1-\nu)/\nu} \lambda^{-M'_k} T_1^{ex,M'_k}(X) \leqq c. \tag{7.2}$$

It follows that for $M'_k \geqq 3$

$$M'_k \geqq \frac{\log T_1^{ex,M'_k}(X) - \log c}{\log \lambda} + \frac{\frac{1-\nu}{\nu} \log\log M'_k}{\log \lambda} \geqq \frac{\log T_1^{ex,M'_k}(X) - \log c}{\log \lambda},$$

and for any $\varepsilon > 0$, there exists a $k_0 \in \mathbb{N}$ such that

$$\log M'_k \geqq (1-\varepsilon) \log\log T_1^{ex,M'_k}(X) \tag{7.3}$$

holds for all $k \geqq k_0$. On the other hand, (7.2) implies

$$|X(T_1^{ex,M'_k}(X))| = 2^{M'_k} \geqq c^{-\nu} (\log M'_k)^{1-\nu} \ (T_1^{ex,M'_k}(X))^{\nu}.$$

This combined with (7.3) leads to

$$\frac{|X(T_1^{ex,M'_k}(X))|}{(T_1^{ex,M'_k}(X))^{\nu} (\log\log T_1^{ex,M'_k}(X))^{1-\nu}} \geqq c^{-\nu}(1-\varepsilon)^{1-\nu},$$

and

$$\varlimsup_{n\to\infty} \frac{|X(n)|}{\psi(n)} \geqq c^{-\nu}(1-\varepsilon)^{1-\nu}.$$

Since ε is arbitrary, we have proved the statement.

□

In the rest of this section we prove (7.1). To this end, first we establish two inequalities. In the following we write T^M for $T_1^{ex,M}$.

By Proposition 6 (2), $G_N^{(i)}(-1)$ converges to $g_i(-1)$ as $N \to \infty$ for $i = 1, 2$, thus there exists a positive constant D such that $G_N^{(i)}(-1) \leqq D$ holds for all $N \in \mathbb{N}$ and $i = 1, 2$. This combined with Chebyshev's inequality gives

$$\hat{P}_N^{(i)}[\ T^N(w) \geqq t\ ] \leqq De^{-\lambda^{-N} t}, \ \ N \in \mathbb{N}, \ \ i = 1, 2, 3, 4$$

where $\hat{P}_N^{(i)}$ is defined in Section 5. In particular, if we set

$$\hat{P}_N^{max}[A] := \max_{i \in \{1,2,3,4\}} \hat{P}_N^{(i)}[A], \quad A \subset \Gamma_N,$$

we have

$$\hat{P}_N^{max}[\ T^N(w) \geqq t\ ] \leqq De^{-\lambda^{-N} t}. \tag{7.4}$$

By Proposition 7 and Theorem 8(1), there exist $C > 0$ such that for any $b > 0$ and $i = 1, 2, 3, 4$,

$$\varliminf_{N\to\infty} {\alpha_N}^{\nu/(1-\nu)} \log \hat{P}_N^{(i)}[\ T^N(w) \leqq \lambda^N \alpha_N\ ] \geqq -C,$$

with $\alpha_N = b\,(\log N)^{-(1-\nu)/\nu}$. This implies that for any $C_0 > C$ there exists $N_0 \in \mathbb{N}$ such that

$$\hat{P}_N^{(i)}[\ T^N(w) \leqq b\lambda^N (\log N)^{-(1-\nu)/\nu}\ ] \geqq N^{-C_0\, b^{-\nu/(1-\nu)}}$$

holds for any $N \geqq N_0$. In particular, if we set

$$\hat{P}_N^{min}[A] := \min_{i \in \{1,2,3,4\}} \hat{P}_N^{(i)}[A], \quad A \subset \Gamma_N,$$

then

$$\hat{P}_N^{min}[\, T^N(w) \leqq b\lambda^N (\log N)^{-(1-\nu)/\nu} \,] \geqq N^{-C_0\, b^{-\nu/(1-\nu)}}, \quad N \geqq N_0. \tag{7.5}$$

Inequalities (7.4) and (7.5) are used to prove Lemma 19 and Lemma 20 below.

Now let $t_n = 2C_0^{(1-\nu)/\nu} \lambda^n (\log n)^{-(1-\nu)/\nu}$ and

$$M_n = \left[\frac{1}{\log \lambda} \left(1 + \frac{2(1-\nu)}{\nu} \right) (n+1) \log\log(n+1) + (\eta+1)n \right],$$

where $[a]$ denotes the largest integer not exceeding a and

$$\eta = \frac{1}{\log \lambda} \ \log \frac{2}{C_0^{(1-\nu)/\nu}}.$$

Lemma 19

$$\sum_{n=1}^{\infty} P[\, T^{M_{n-1}}(X) > \frac{1}{2}\ t_{M_n} \,] < \infty.$$

Proof. By the definition of M_n, we have for $n \geqq 2$,

$$M_n - M_{n-1} \geqq \frac{1}{\log \lambda} \left(1 + \frac{2(1-\nu)}{\nu} \right) \log\log n + \eta.$$

It follows that

$$\lambda^{-M_{n-1}} t_{M_n}/2 = C_0^{(1-\nu)/\nu} \lambda^{M_n - M_{n-1}} (\log M_n)^{-(1-\nu)/\nu} \geqq 2 \log n \left(\frac{(\log n)^2}{\log M_n} \right)^{(1-\nu)/\nu}.$$

Since $M_n = O(n \log\log n)$, $\lambda^{-M_{n-1}} t_{M_n}/2 \geqq 2\log n$ holds for n sufficiently large. This combined with (7.4) gives

$$P[\, T^{M_{n-1}}(X) > \frac{1}{2} t_{M_n} \,] \leqq \hat{P}_{M_{n-1}}^{max}[\, T^{M_{n-1}}(w) \geqq \frac{1}{2} t_{M_n} \,] \leqq De^{-\lambda^{-M_{n-1}}\, t_{M_n}/2} \leqq De^{-2\log n} = \frac{D}{n^2},$$

which implies the desired convergence. □

Lemma 20

$$\sum_{n=1}^{\infty} \hat{P}_{M_n}^{min}[\, T^{M_n}(w) - T^{M_{n-1}}(w) \leqq \frac{1}{2}\ t_{M_n} \,] = \infty.$$

Proof. This follows from (7.5). Setting $b = C_0^{(1-\nu)/\nu}$ in (7.5) and by the definition of M_n, we have for n_0 sufficiently large,

$$\sum_{n=1}^{\infty} \hat{P}_{M_n}^{min}[\, T^{M_n}(X) - T^{M_{n-1}}(X) \leqq \frac{1}{2} t_{M_n}] \geqq \sum_{n=1}^{\infty} \hat{P}_{M_n}^{min}[\, T^{M_n}(w) \leqq \frac{1}{2} t_{M_n} \,] \geqq \sum_{n=n_0}^{\infty} M_n^{-1}$$

$$\geqq \frac{1}{\eta + 2 + \frac{1}{\log \lambda}(1 + 2\frac{1-\nu}{\nu})} \sum_{n=n_0+1}^{\infty} \frac{1}{n \log\log n} \geqq c_1 \sum_{n=n_0+1}^{\infty} \frac{1}{n \log n} = \infty,$$

where c_1 is a positive constant. □

Now we go on to show that (7.1) holds. We see that for any $k, K \in \mathbb{N}$ with $k < K$

$$\begin{aligned} & P[\bigcup_{\ell=k}^{K}\{T^{M_\ell}(X) \leqq t_{M_\ell}\}] \geqq P[\bigcup_{\ell=k}^{K}\{T^{M_\ell}(X) - T^{M_{\ell-1}}(X) \leqq \frac{1}{2}t_{M_\ell},\ T^{M_{\ell-1}}(X) \leqq \frac{1}{2}t_{M_\ell}\}] \\ & = 1 - P[\bigcap_{\ell=k}^{K}(\{T^{M_\ell}(X) - T^{M_{\ell-1}}(X) > \frac{1}{2}t_{M_\ell}\} \cup \{T^{M_{\ell-1}}(X) > \frac{1}{2}t_{M_\ell}\})] \qquad (7.6) \\ & \geqq 1 - P[\bigcap_{\ell=k}^{K}\{T^{M_\ell}(X) - T^{M_{\ell-1}}(X) > \frac{1}{2}t_{M_\ell}\}] - \sum_{\ell=k}^{K} P[T^{M_{\ell-1}}(X) > \frac{1}{2}t_{M_\ell}]. \end{aligned}$$

We first show that

$$P[\bigcap_{\ell=k}^{K}\{T^{M_\ell}(X) - T^{M_{\ell-1}}(X) > \frac{1}{2}t_{M_\ell}\}] \to 0, \quad K \to \infty$$

holds.

Note that by definition

$$\begin{aligned} & P[\bigcap_{\ell=k}^{K}\{T^{M_\ell}(X) - T^{M_{\ell-1}}(X) > \frac{1}{2}t_{M_\ell}\}] \\ & = \tilde{P}_{M_K}[\bigcap_{\ell=k}^{K}\{T^{M_\ell}(w) - T^{M_{\ell-1}}(w) > \frac{1}{2}t_{M_\ell}\}] \quad \text{(by (5.4) and (5.7))} \\ & = P^{rw}_{M_K}[\bigcap_{\ell=k}^{K}\{T^{M_\ell}(Lw) - T^{M_{\ell-1}}(Lw) > \frac{1}{2}t_{M_\ell}\}] \quad \text{(by (5.2))} \\ & = \frac{11}{28}(P^{(1)}_{M_K} + P^{(2)}_{M_K})[\bigcap_{\ell=k}^{K}\{T^{M_\ell}(Lw) - T^{M_{\ell-1}}(Lw) > \frac{1}{2}t_{M_\ell}\}] \\ & + \frac{3}{28}(P^{(3)}_{M_K} + P^{(4)}_{M_K})[\bigcap_{\ell=k}^{K}\{T^{M_\ell}(Lw) - T^{M_{\ell-1}}(Lw) > \frac{1}{2}t_{M_\ell}\}] \text{ (by (5.3))} \end{aligned}$$

Now recall the procedure for loop erasure. Note that $M_k < M_K$. In the process of erasing 2^{M_k}–scale loops from the random walk $\tilde{Z}_{M_K}$, we obtain $\hat{Q}_{M_k}\tilde{Z}_{M_K}$. ($\hat{Q}_M$ is defined at the end of Section 3. Recall that for $M < N$ and $w \in W_N \cup V_N$, $\hat{Q}_M w$ is a loopless path on the coarse graph $2^M F_0$.) Then we restore the original fine structures to each step of $\hat{Q}_{M_k}\tilde{Z}_{M_K}$ and continue loop erasure. For each Δ_i in $\sigma_{M_k}(\hat{Q}_{M_k}\tilde{Z}_{M_K})$, if Δ_i is Type 1 with regard to $\hat{Q}_{M_k}\tilde{Z}_{M_K}$, the rest of the procedure is the same as loop erasure for Z_{M_k} (modulo rotation and reflection), and if Type 2, the same as that for Z'_{M_k}(Z_M and Z'_M are defined in (2.1) and (2.2)). Conditioned on $\hat{Q}_{M_k}\tilde{Z}_{M_K}$, path segments in different 2^{M_k}–triangles are independent. Thus

$$P^{(1)}_{M_K}[\ \bigcap_{\ell=k}^{K}\{T^{M_\ell}(Lw)-T^{M_{\ell-1}}(Lw)>\frac{1}{2}t_{M_\ell}\}\ ]$$
$$=\sum_{w'\in\Gamma_{M_K-M_k}} P^{(1)}_{M_K}[\ \bigcap_{\ell=k}^{K}\{T^{M_\ell}(Lw)-T^{M_{\ell-1}}(Lw)>\frac{1}{2}t_{M_\ell}\}\mid 2^{-M_k}\hat{Q}_{M_k}(w)=w'\ ]$$
$$\times P^{(1)}_{M_K}[\ 2^{-M_k}\hat{Q}_{M_k}(w)=w'\ ]$$

$$=\sum_{w'\in\Gamma_{M_K-M_k}} P^{(1)}_{M_K}[\ T^{M_k}(Lw)-T^{M_{k-1}}(Lw)>\frac{1}{2}t_{M_k}\mid 2^{-M_k}\hat{Q}_{M_k}(w)=w'\ ]$$
$$\times P^{(1)}_{M_K}[\ \bigcap_{\ell=k+1}^{K}\{T^{M_\ell}(Lw)-T^{M_{\ell-1}}(Lw)>\frac{1}{2}t_{M_\ell}\}\mid 2^{-M_k}\hat{Q}_{M_k}(w)=w'\ ]P^{(1)}_{M_K}[2^{-M_k}\hat{Q}_{M_k}(w)=w']$$

$$=\sum_{w'\in\Gamma_{M_K-M_k}} \hat{P}^{(*)}_{M_k}[\ T^{M_k}(w)-T^{M_{k-1}}(w)>\frac{1}{2}t_{M_k}\ ]$$
$$\times P^{(1)}_{M_K}[\ \bigcap_{\ell=k+1}^{K}\{T^{M_\ell}(Lw)-T^{M_{\ell-1}}(Lw)>\frac{1}{2}t_{M_\ell}\}\mid 2^{-M_k}\hat{Q}_{m_k}(w)=w'\ ]P^{(1)}_{M_K}[2^{-M_k}\hat{Q}_{M_k}(w)=w']$$
$$\leqq \hat{P}^{max}_{M_k}[\ T^{M_k}(w)-T^{M_{k-1}}(w)>\frac{1}{2}t_{M_k}\ ]$$
$$\times\sum_{w'\in\Gamma_{M_K-M_k}} P^{(1)}_{M_K}[\ \bigcap_{\ell=k+1}^{K}\{T^{M_\ell}(Lw)-T^{M_{\ell-1}}(Lw)>\frac{1}{2}t_{M_\ell}\}\mid 2^{-M_k}\hat{Q}_{m_k}(w)=w'\ ]$$

$$\times P^{(1)}_{M_K}[\ 2^{-M_k}\hat{Q}_{M_k}(w)=w'\ ]$$
$$\leqq \hat{P}^{max}_{M_k}[\ T^{M_k}(w)-T^{M_{k-1}}(w)>\frac{1}{2}t_{M_k}\ ]\ P^{(1)}_{M_K}[\ \bigcap_{\ell=k+1}^{K}\{T^{M_\ell}(Lw)-T^{M_{\ell-1}}(Lw)>\frac{1}{2}t_{M_\ell}\}],$$

where $\hat{P}^{(*)}_{M_k}=P^{(1)}_{M_k}$ if the first element of $\sigma_{M_k}(\hat{Q}_{M_k}w)$ is Type 1, and $\hat{P}^{(*)}_{M_k}=P^{(2)}_{M_k}$ if Type 2.

We have similar results for $i=2,3,4$ to have

$$P^{rw}_{M_K}[\ \bigcap_{\ell=k}^{K}\{T^{M_\ell}(Lw)-T^{M_{\ell-1}}(Lw)>\frac{1}{2}t_{M_\ell}\}\ ]$$
$$\leqq \hat{P}^{max}_{M_k}[\ T^{M_k}(w)-T^{M_{k-1}}(w)>\frac{1}{2}t_{M_k}\ ]\ P^{rw}_{M_K}[\ \bigcap_{\ell=k+1}^{K}\{T^{M_\ell}(Lw)-T^{M_{\ell-1}}(Lw)>\frac{1}{2}t_{M_\ell}\}\ ].$$

Repeating this procedure, we have

$$
\begin{aligned}
&P[\bigcap_{\ell=k}^{K}\{T^{M_\ell}(X) - T^{M_{\ell-1}}(X) > \frac{1}{2}t_{M_\ell}\}\ ] \\
&= P^{rw}_{M_K}[\ \bigcap_{\ell=k}^{K}\{T^{M_\ell}(Lw) - T^{M_{\ell-1}}(Lw) > \frac{1}{2}t_{M_\ell}\}\ ] \\
&\leqq \prod_{\ell=k}^{K}\hat{P}^{max}_{M_\ell}[\ T^{M_\ell}(w) - T^{M_{\ell-1}}(w) > \frac{1}{2}t_{M_\ell}\ ] \\
&= \prod_{\ell=k}^{K}(1 - \hat{P}^{min}_{M_\ell}[\ T^{K_\ell}(w) - T^{M_{\ell-1}}(w) \leqq \frac{1}{2}t_{M_\ell}\ ]) \\
&\leqq \exp(-\sum_{\ell=k}^{K}\hat{P}^{min}_{M_\ell}[\ T^{M_\ell}(w) - T^{M_{\ell-1}}(w) \leqq \frac{1}{2}t_{M_\ell}\ ]) \\
&\to 0 \quad (K \to \infty),
\end{aligned}
$$

where we used Lemma 20.

By Lemma 19, we see that the third term in the rightmost side in (7.6) converges to 0, first taking the limit as $K \to \infty$, then $k \to \infty$.

Thus

$$
\begin{aligned}
&P[\bigcap_{k=1}^{\infty}\bigcup_{M=k}^{\infty}\{(\log M)^{(1-\nu)/\nu}\lambda^{-M}T_1^{ex,M}(X) \leqq 2C_0^{(1-\nu)/\nu}\}\ ] = P[\bigcap_{k=1}^{\infty}\bigcup_{\ell=k}^{\infty}\{T^{M_\ell}(X) \leqq t_{M_\ell}\}\ ] \\
&= \lim_{k\to\infty}\lim_{K\to\infty} P[\bigcup_{\ell=k}^{K}\{T^{M_\ell}(X) \leqq t_{M_\ell}\}\ ] = 1,
\end{aligned}
$$

which proves (7.1).

Proposition 16 combined with Proposition 17 completes the proof of Theorem 3.

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