Non-Markov processes on fractals

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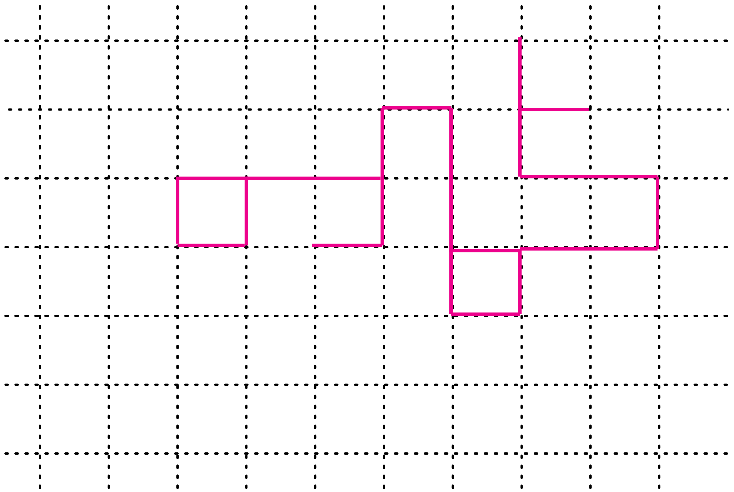
3rd Bremen Winter School and Symposium
Diffusion on Fractals and Non-linear Dynamics

Markov vs. Non-Markov

Markov

ex. Simple random walk

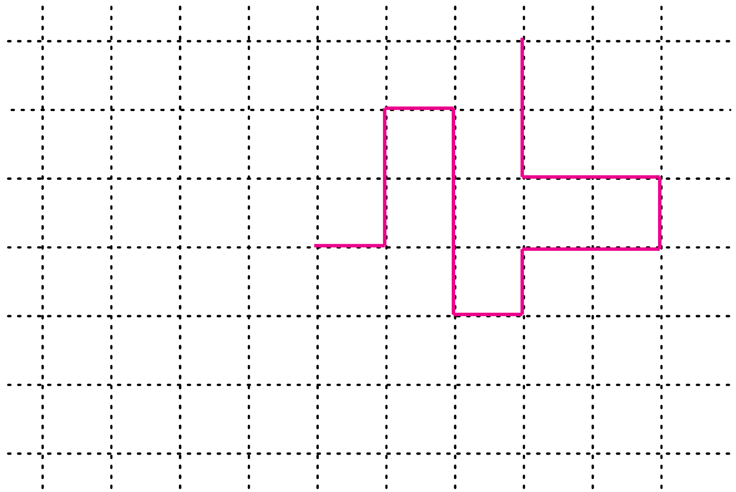
Jumps to one of the nearest sites with equal probability.



non-Markov

ex. Self-avoiding walk

Cannot visit any sites more than once.



We focus on two basic questions:

(1) Displacement exponent

(2) Scaling limit

for the following processes on a fractal (SG):

1. Self-avoiding walks (SAW)
2. Loop-erased random walks (LERW)
3. Self-repelling walks (and their loop-erasure)

in terms of

(A) Find a model that suits fractals

(B) Extract information on ‘standard’ models.

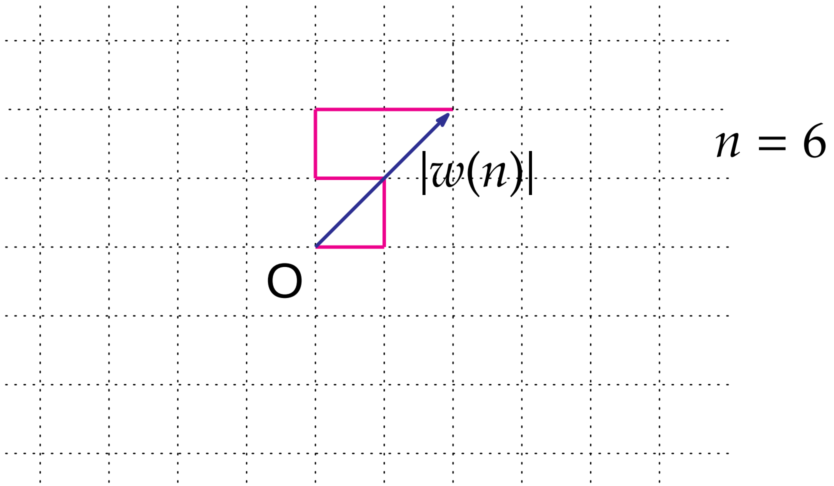
Outline

1. Self-avoiding walk
   1-1. Two basic questions
   1-2. Background
   1-3. Fixed-ends model and answers to the questions
   1-4. Generating functions and recursions
2. Loop-erased random walk
   2-1. Erasing-larger-loops-first model
   2-2. Answers to the questions
3. Self-repelling walks (and their loop-erasure)

1. Self-avoiding walk

(1) How far can an n-step walk go in average?

For each fixed n, consider the set of all n-step self-avoiding paths starting from O, and assign equal probability to each n-step path $\rightarrow$ 'standard' self-avoiding walk. $w(n)$: the location after n-steps, $|w(n)|$: Euclidian distance from O.



Mean square displacement $E[|w(n)|^2] \sim \ ? \quad n \rightarrow \infty$

1-1. Two basic questions

(1) How far can an n-step walk go in average?

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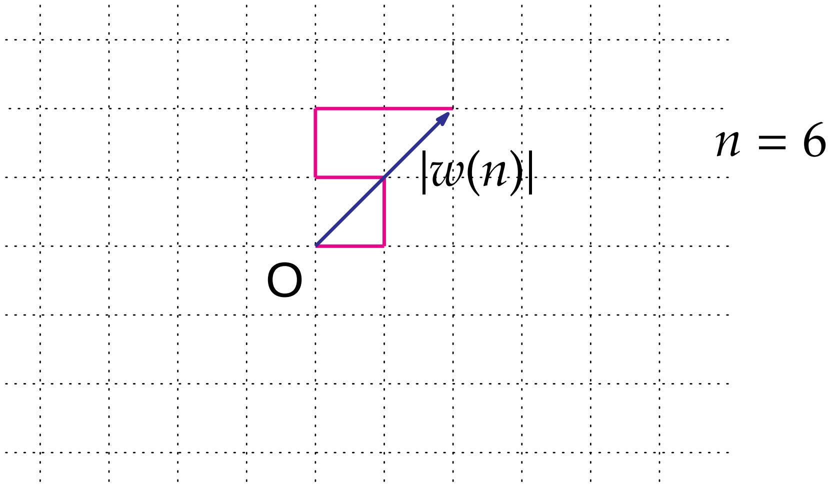


Mean square displacement $E[|w(n)|^2] \sim \ ? \quad n \rightarrow \infty$

If the mean square displacement shows a power behavior like $E[|w(n)|^2] \sim n^{2\nu}, \quad n \to \infty$,
ν : the displacement exponent .

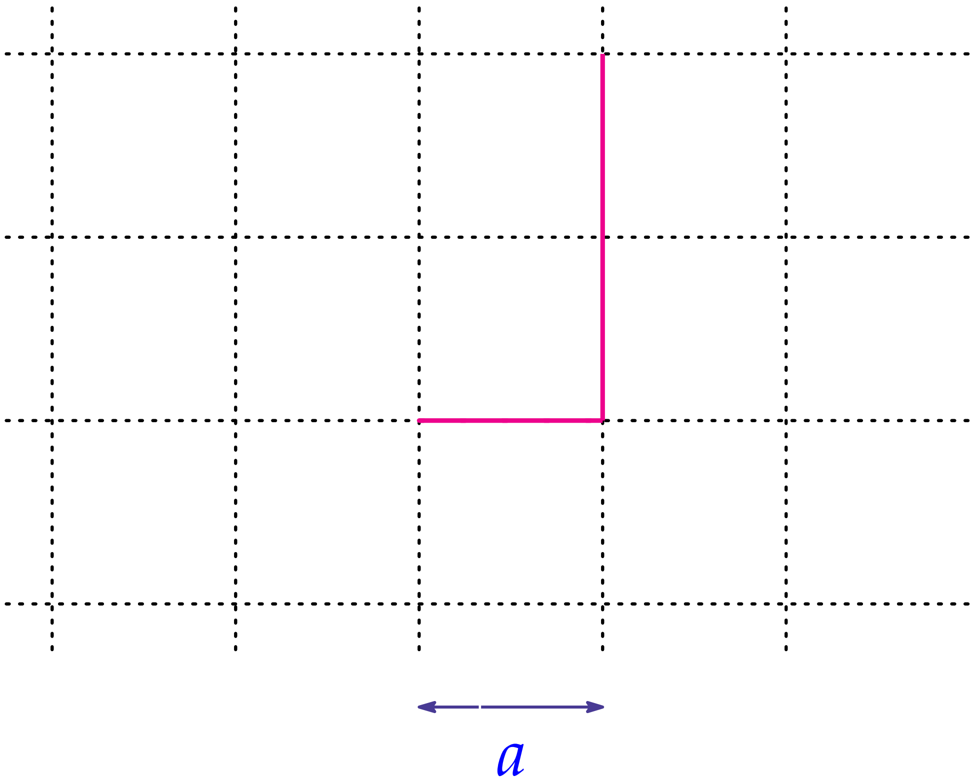
cf. Simple random walk on $\mathbb{Z}^d$

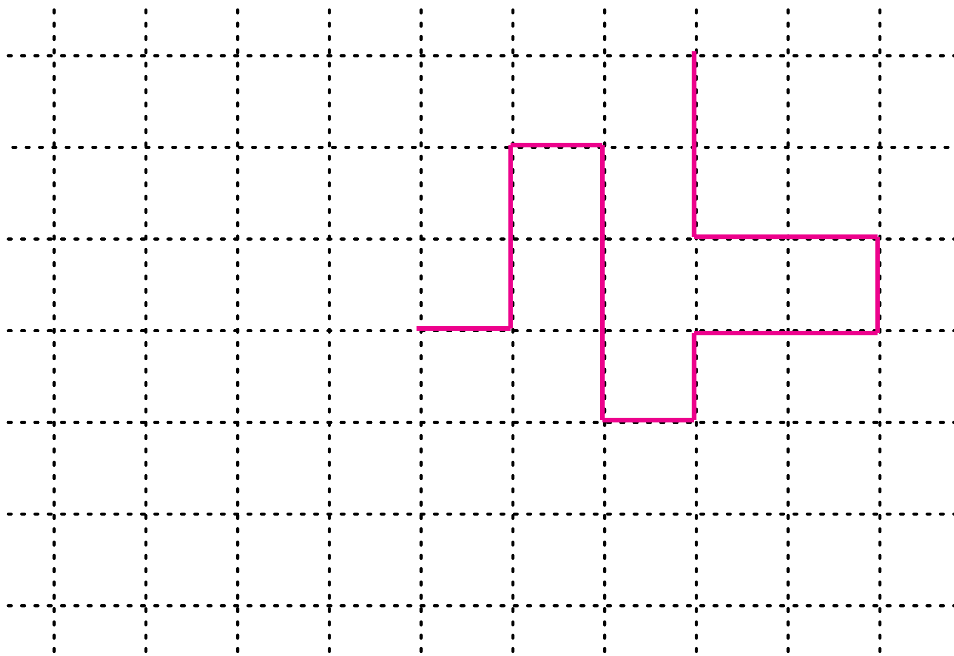
$$E[|w(n)|^2] = n, \ \nu = 1/2.$$

(2) Scaling limit (The limit as the edge length $a \to 0$)

Does the scaling limit exist? (Does the SAW converge to any limit process as $a \to 0$?)

If yes, what is the limit process like?





cf. Simple random walk on $(a\mathbb{Z})^d \to d$-dimensional BM

$(a \to 0)$

1-2. Background

SAW on $\mathbb{Z}^d$

displacement exponent		scaling limit
$d = 1$	$\nu = 1$	trivial
$d = 2$	$\nu = \frac{3}{4}$	$\mathrm{SLE}_{8/3}$
$d = 3$	$\nu = 0.5876\cdots$	?
$d = 4$	$\nu = \frac{1}{2}$+(log correction)	BM
$d \geq 5$	$\nu = \frac{1}{2}$	BM (Hara, Slade)

Low dimensions are tough! blue : conjectures.

$\Longrightarrow$ What about SAW on fractals?

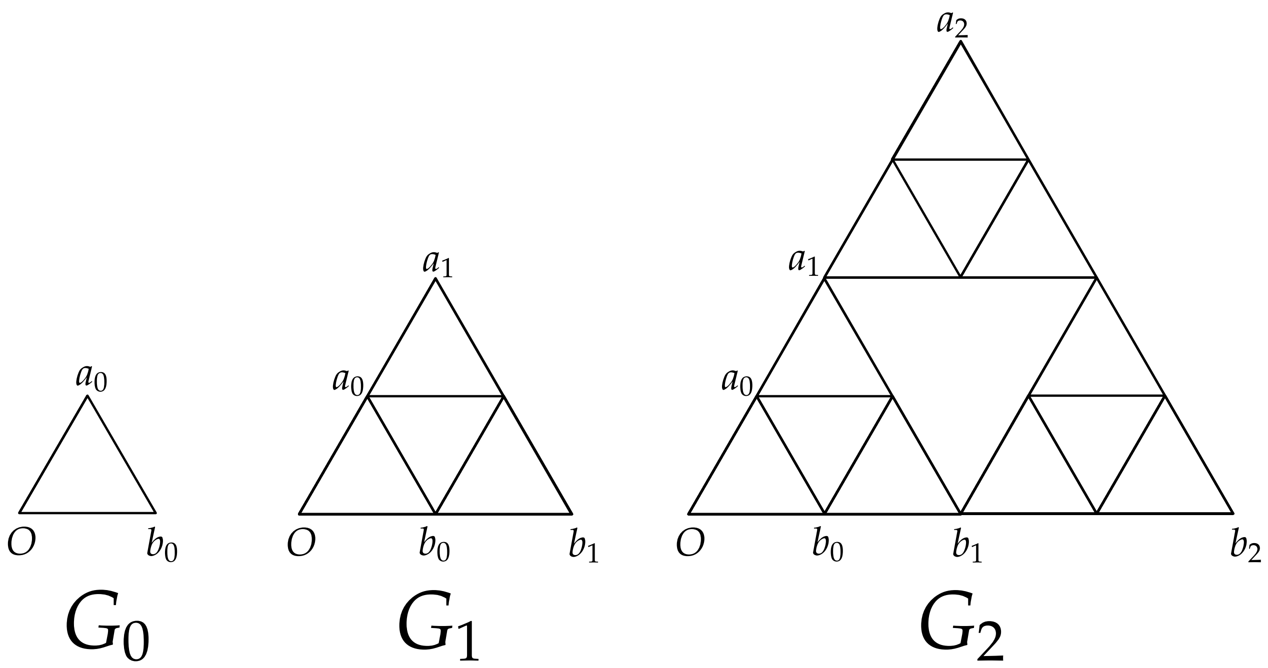
SAW on $\mathbb{Z}^d$

displacement exponent		scaling limit
$d = 1$	$\nu = 1$	trivial
$d_H = 1.58$	Sierpinski gasket	
$d = 2$	$\nu = \frac{3}{4}$	$\mathrm{SLE}_{8/3}$
$d = 3$	$\nu = 0.5876\cdots$	?
$d = 4$	$\nu = \frac{1}{2}$+(log correction)	BM
$d \geq 5$	$\nu = \frac{1}{2}$	BM (Hara, Slade)

Low dimensions are tough! blue : conjecture.

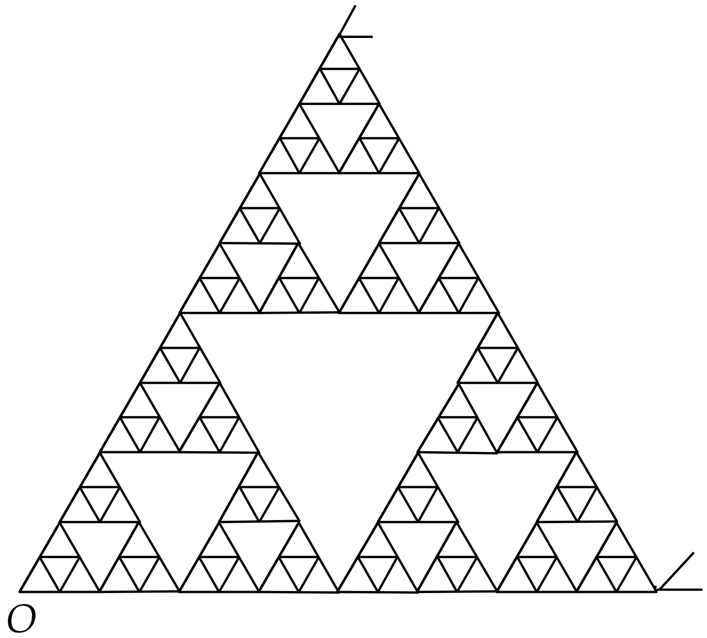
Pre-Sierpinski gasket

G_0: a unit triangle, $G_N \times 3 \to G_{N+1}$



$a_N = 2^N a_0$, $b_N = 2^N b_0$, $\triangle O a_N b_N$: the outer triangle of G_N.

Infinite iteration → an infinite graph



$G_\infty = \cup_{N=1}^{\infty} G_N$: the pre-Sierpinski Gasket

an infinite graph with edge length 1.

(1) Displacement exponent

2-dim Sierpinski gasket $\left(d_H = \dfrac{\log 3}{\log 2} = 1.58\ldots\right)$

Physicists had known the answer. (1970's and 1980's)

$$\nu = \frac{\log 2}{\log \lambda} = 0.798\ldots > \frac{1}{2}, \quad \lambda = \frac{7-\sqrt{5}}{2} = 2.38\cdots.$$

Mathematicians proved the answer. (1990's)

ν exists and the above answer is right.

The scaling limit exists.

To solve mathematically:

(A) Construct a model that leads to recursion relations of generating functions (making use of fractal structures).

(B) Extract information for the ‘standard’ SAW (uniform distribution on n-step paths).

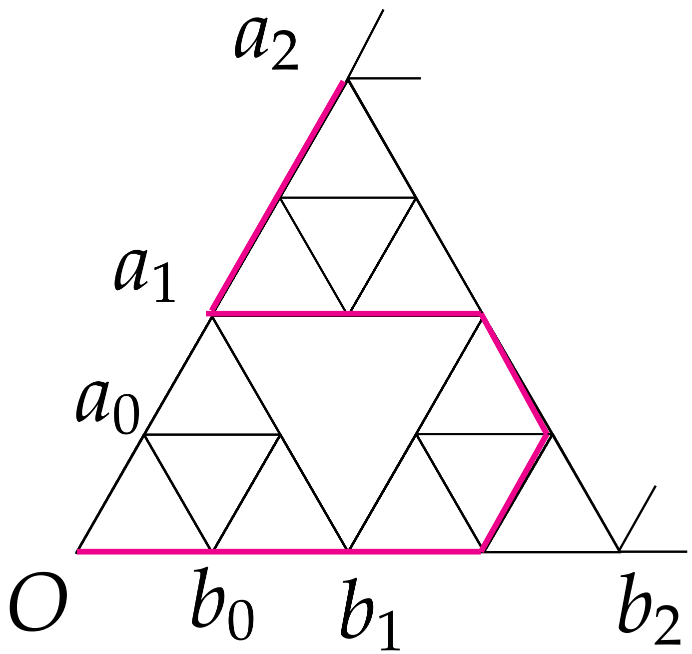
1.-3. Fixed-ends model and answers

$\triangle Oa_Nb_N$: the outer triangle of G_N. For each N,

W_N : the set of all self-avoiding paths $O \to a_N$ in G_N.

$L(w) = \sharp$ (steps of path w), Fix $\beta > 0$: parameter.

Assign each $w \in W_N$ probability $P_N[w] \propto e^{-\beta L(w)}$



$N = 2$

$w \in W_2 = \{\text{paths } O \to a_2\}$

$L(w) = 9$

$P[w] \propto e^{-9\beta}$

Natural in two ways.

Thm. 1 Displacement exponent (T. Hattori, Kusuoka 1992)

For the 'standard' SAW (equal prob. to each n-step path),

$$\forall s > 0, \ \lim_{n\to\infty} \frac{\log E[|w(n)|^s]}{\log n} = s\nu, \ \nu = \frac{\log 2}{\log \lambda} = 0.798\cdots > 1/2.$$

$$\lambda = 2x_c + 3x_c^2$$

(B) This result is obtained by looking into the behavior of the dynamical system near the fixed point.

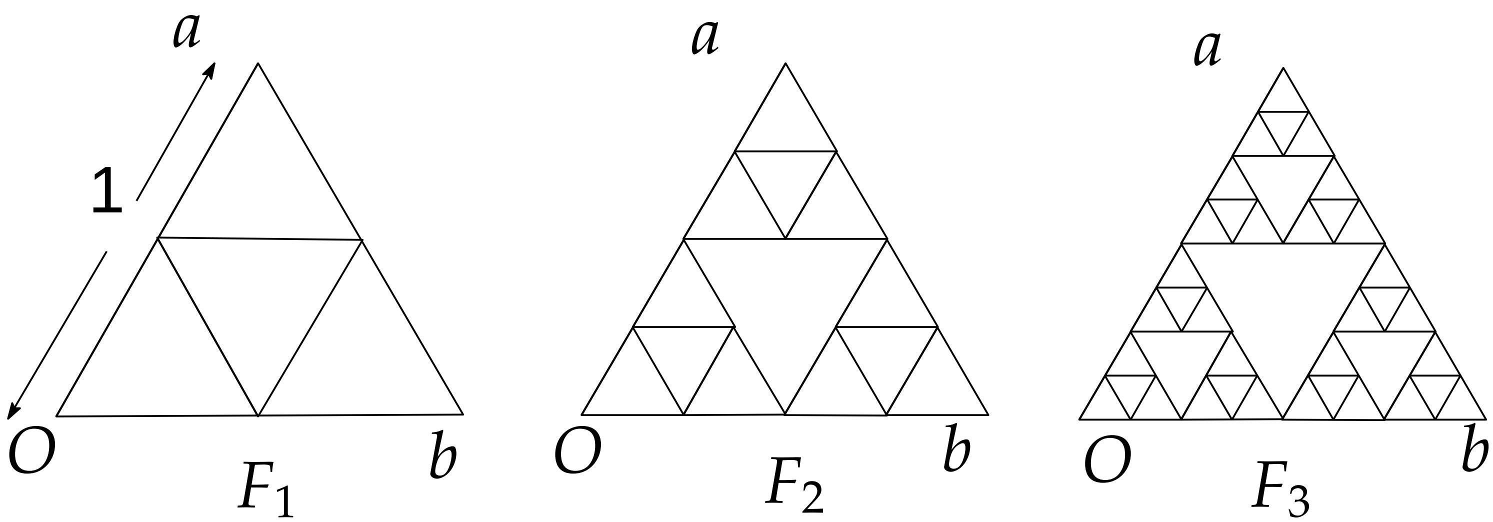
$N_n = \sharp\{$n-step SA paths$\}$, $\beta_c = 0.8276\cdots$,

$\exists C, C', \gamma, \gamma' > 0$;

$$Cn^{-\gamma} e^{\beta_c n} \le N_n \le C' n^{\gamma'} e^{\beta_c n}$$

(2) The scaling limit.
Consider finite pre-Sierpinski gaskets.

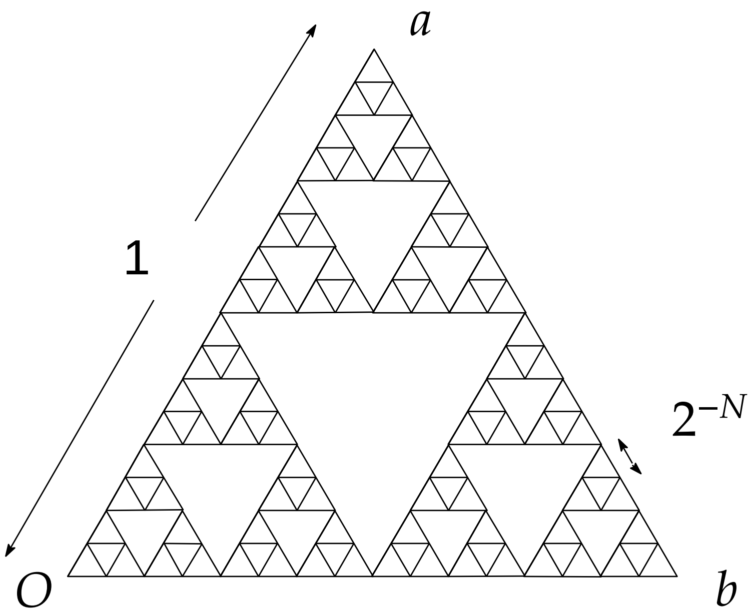
$\triangle Oab$: a unit triangle. $F_N = 2^{-N} G_N$ a graph with edge length 2^{-N}. Sierpinski gasket $F = \cup_{N=1}^{\infty} F_N$



F_N : pre-SG with edge length 2^{-N}

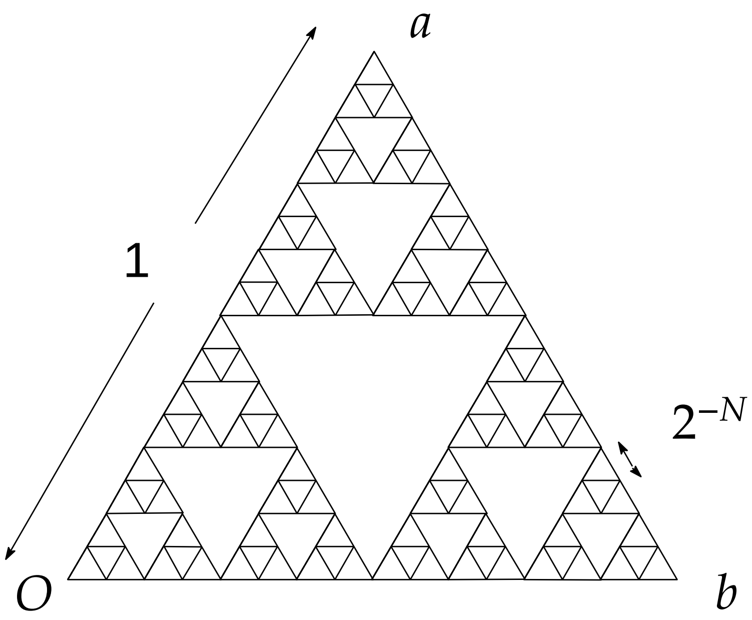
Shrink the fixed-end SAW by 2^{-N}.(step size 2^{-N}).

$X_N(i)$: the location of SAW (from O to a) at the i-th step.



For $w = (w_0, w_1, w_2, \cdots, w_{L(w)}) \in 2^{-N} W_N$,

$$P[X_N(i) = w_i,\ i = 1, 2, \cdots, L(w)] \propto e^{-\beta L(w)}$$



reminder : $e^{-\beta L(w)}$

smaller steps → needs acceleration

Thm. 2 Scaling limit (T. Hattori, K.H. 1991)

As $N \to \infty$

$\beta > \beta_c$ $X_N(\, 2^N t\,) \to$ 【constant motion along $\overline{Oa}$】

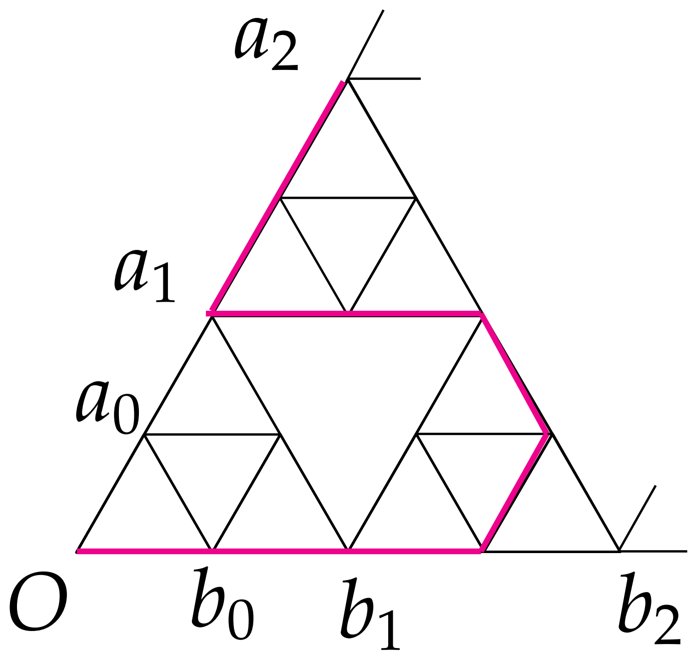
$\beta = \beta_c$ $X_N(\, \lambda^N t\,) \to$ 【Self-avoiding process】

$d_H (= 1/\nu) > 1$ a.s. (ν: displ. exp.)

$\beta < \beta_c$ $X_N(\, 3^N t\,) \to$ 【Peano curve】

1-4. Generating functions and recursions

Going back to the pre-Sierpinski gasket with edge length 1,



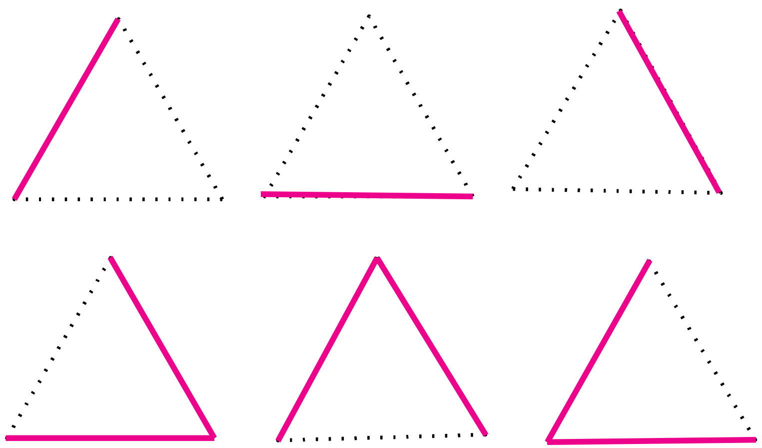
$N = 2$

$w \in W_2 = \{\text{paths } O \to a_2\}$

$L(w) = 9$

$P[w] \propto e^{-9\beta}$

Preparation for the definition of generating functions :
For a path $w \in W_N$, count the numbers of unit triangles w passes through:



Type 1

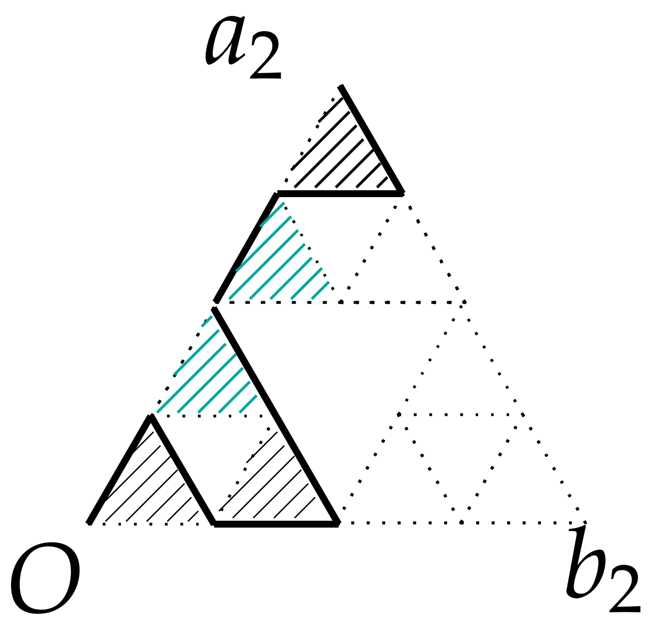
Type 2

$$s_1(w) = \sharp\{\text{triangles of Type 1}\}$$

$$s_2(w) = \sharp\{\text{triangles of Type 2}\}$$

Random variables

$$s_1(w) = \sharp\{\text{Type 1}\}$$
$$s_2(w) = \sharp\{\text{Type 2}\}$$



$$s_1(w) = 2, \; s_2(w) = 3$$

Number of steps : $L(w) = s_1(w) + 2s_2(w)$

(In other words, ‘time’ it takes to go $O \to a$ if jumps occur at integer times.)

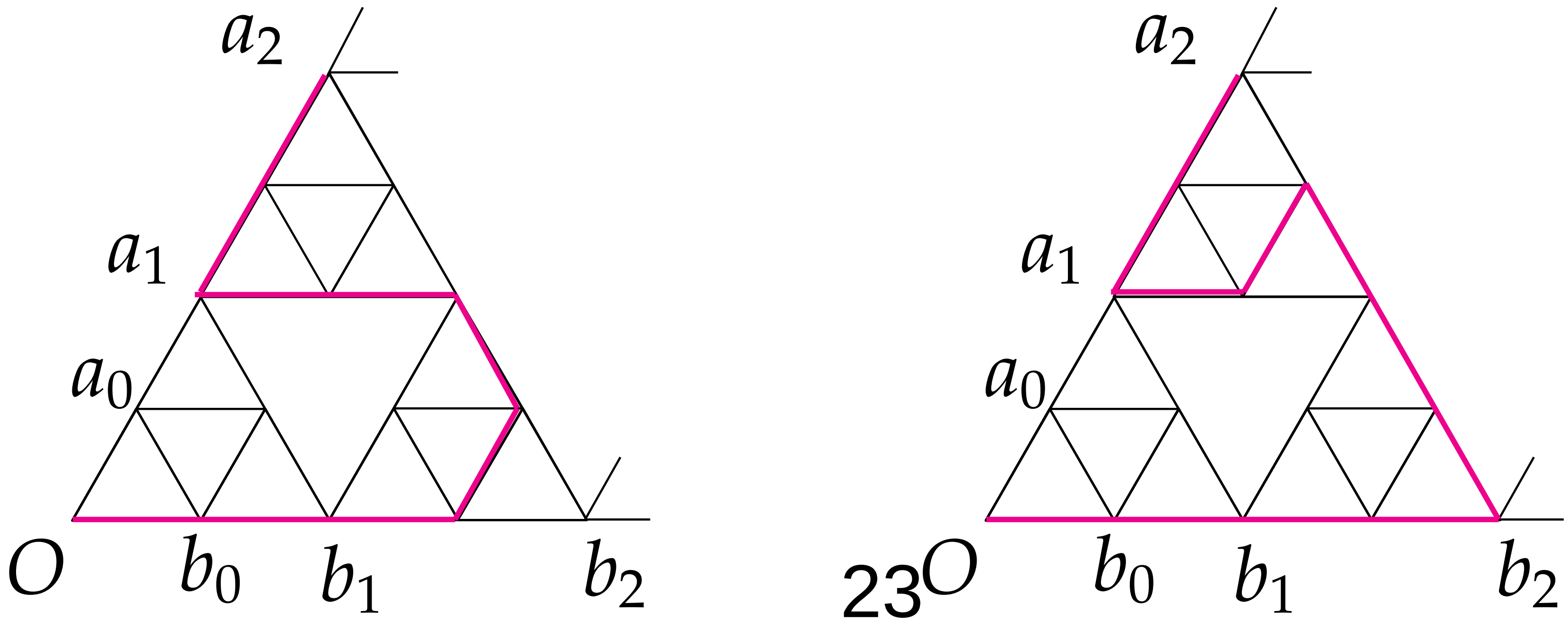
Genarating functions

$W_N = W_{1,N} \cup W_{2,N}$, $\quad x, y \geq 0$

$W_{1,N}$: Paths not visiting b_N, $W_{2,N}$: Paths visiting b_N,

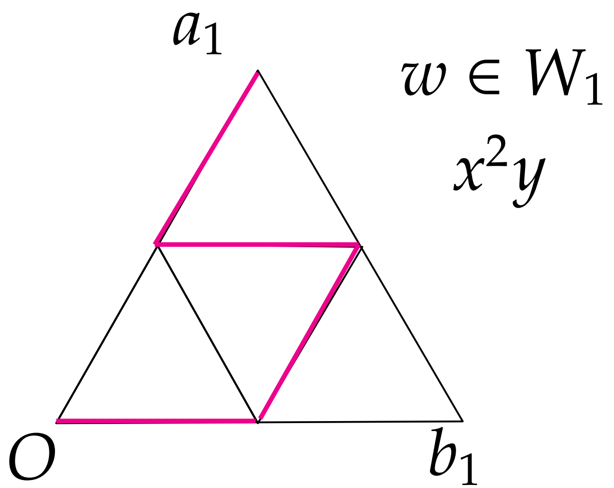
$x, y \geq 0$

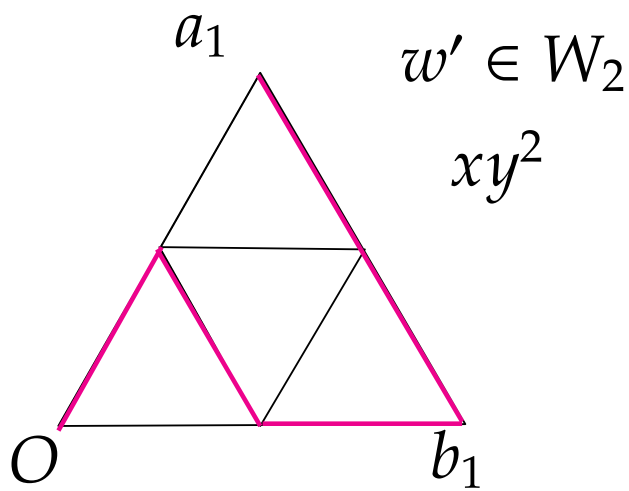
$$\Phi_N(x,y) = \sum_{w \in W_{1,N}} x^{s_1(w)} y^{s_2(w)}, \quad \Theta_N(x,y) = \sum_{w \in W_{2,N}} x^{s_1(w)} y^{s_2(w)}.$$



$$\Phi_1(x, y) = x^2 + 2xy + y^2 + 2x^2y + x^3,$$

$$\Theta_1(x, y) = x^2y + 2xy^2, \quad x, y \geq 0$$

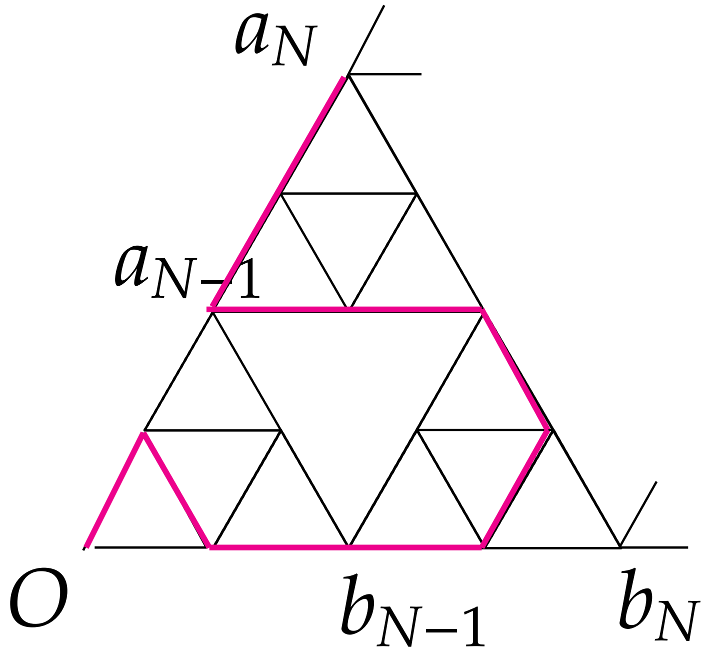


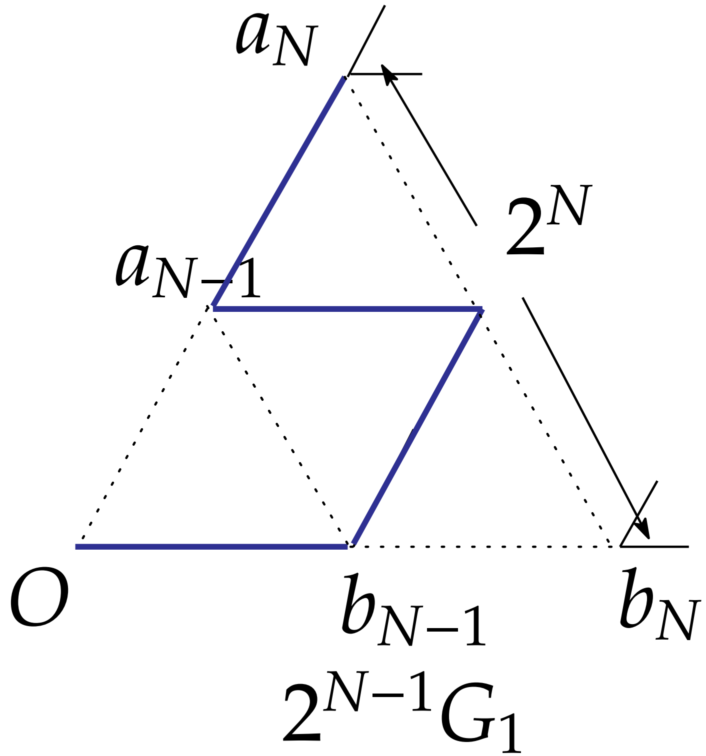


Recursions

$$(\Phi_N(x,y),\Theta_N(x,y)) =$$

$$(\Phi_1(\Phi_{N-1}(x,y),\Theta_{N-1}(x,y)),\Theta_1(\Phi_{N-1}(x,y),\Theta_{N-1}(x,y))).$$

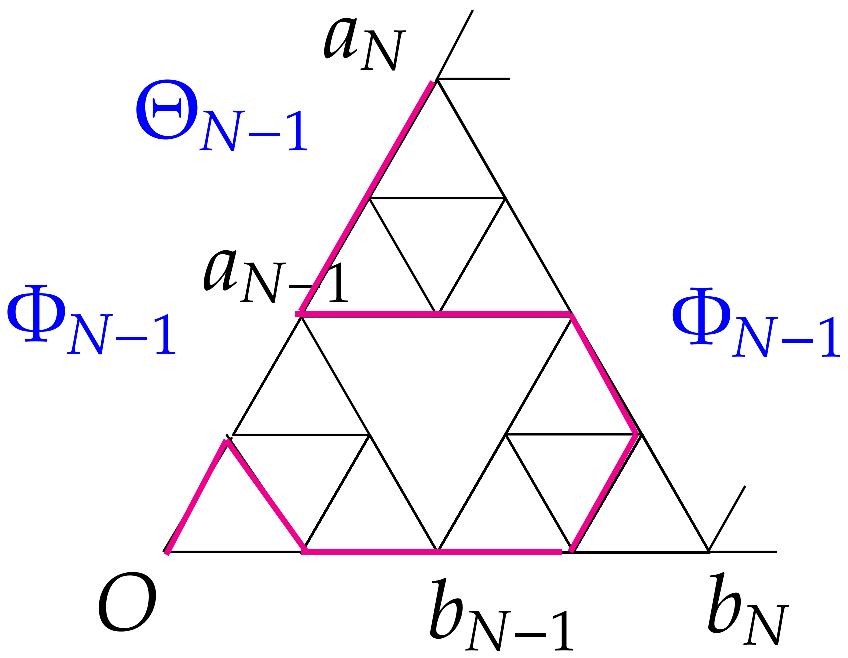


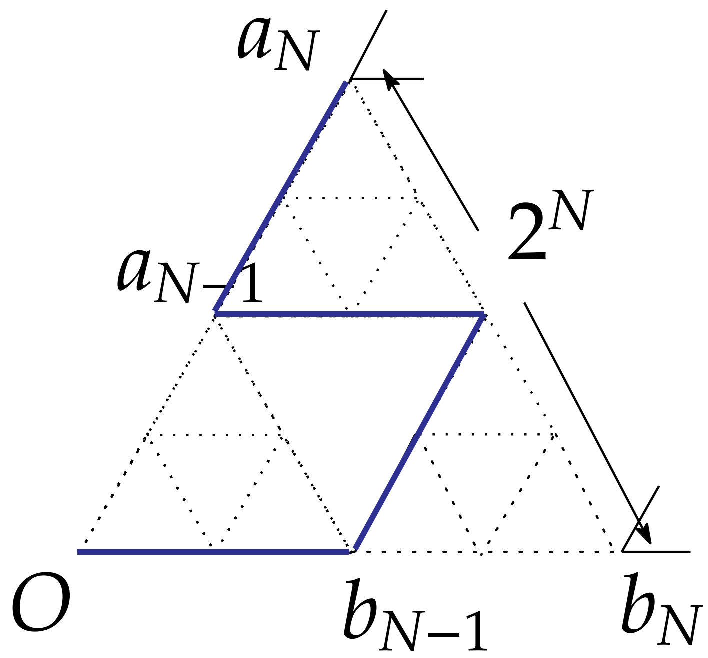


$$\Phi_N(x,y) := \sum_{w \in W_{1,N}} x^{s_1(w)} y^{s_2(w)} = \Phi_1(\Phi_{N-1}(x,y), \Theta_{N-1}(x,y)).$$

Decompose $w \in W_N$ into a coarse path and finer structures.

Blue : 2^{N-1}-scale coarse paths (similar to a path in W_1) $\to x^2 y$.





$$\Phi_N(x,y) := \sum_{w \in W_{1,N}} x^{s_1(w)} y^{s_2(w)} = \Phi_1(\Phi_{N-1}(x,y), \Theta_{N-1}(x,y)).$$

Each 2^{N-1}- triangle is congruent to $G_{N-1} \to \Phi_{N-1}^2 \Theta_{N-1}$.

Recursions (Obtained from fractal structure)

$$(\Phi_N(x,y),\Theta_N(x,y)) =$$

$$(\Phi_1(\Phi_{N-1}(x,y),\Theta_{N-1}(x,y)),\Theta_1(\Phi_{N-1}(x,y),\Theta_{N-1}(x,y))).$$

Two-dimentional dynamical system.

Iterations of (Φ_1,Θ_1)

$(x,y) \to (\Phi_1(x,y),\Theta_1(x,y)) \to (\Phi_2(x,y),\Theta_2(x,y)) \to \cdots$

$$\Phi_1(x,y) = x^2 + 2xy + y^2 + 2x^2y + x^3.$$

Asymptotic behavior of the dynamical system.

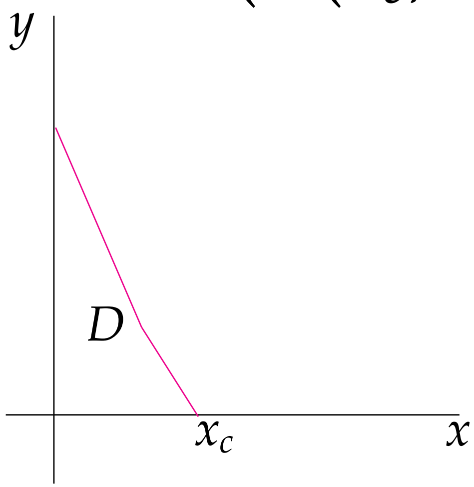
$\exists\, D \in \mathbb{R}_+^2$ open

As $N \to \infty$,

$$(\Phi_N(x,y), \Theta_N(x,y)) \to \begin{cases} (0,0), & (x,y) \in D \\ (x_c, 0), & (x,y) \in \partial D \\ (\infty, \infty), & (x,y) \in \mathbb{R}_+^2 \setminus \overline{D}. \end{cases}$$

$(x_c, 0)$: the unique fixed point in $\mathbb{R}_+^2 \setminus \{(0,0)\}$.

$$(\Phi(x_c,0), \Theta(x_c,0)) = (x_c, 0), \quad x_c = (\sqrt{5} - 1)/2$$



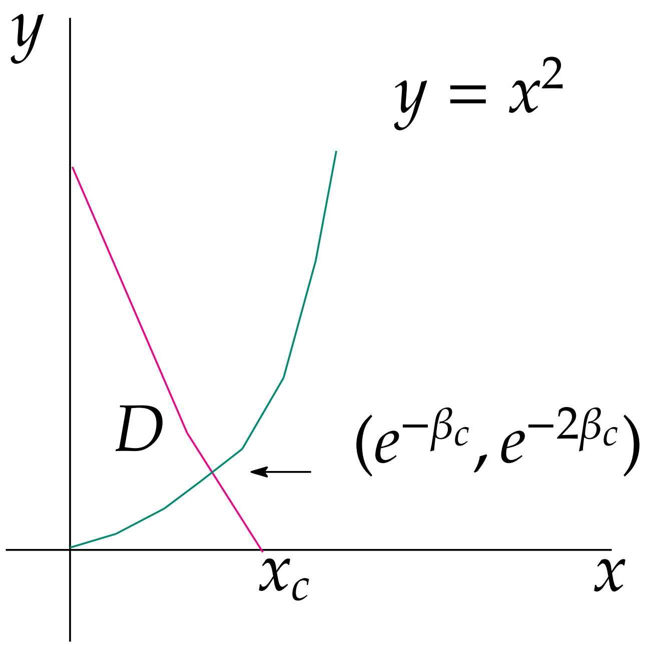
Fixed-ends model

A special choice for (x, y) gives the fixed-ends model.

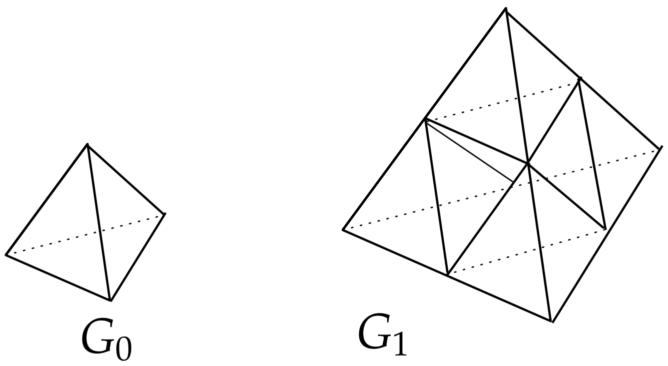
$$\Phi_N(e^{-\beta}, e^{-2\beta}) = \sum_{w \in W_{1,N}} e^{-\beta L(w)}, \ \Theta_N(e^{-\beta}, e^{-2\beta}) = \sum_{w \in W_{2,N}} e^{-\beta L(w)}$$

$$\exists \beta_c; \ (e^{-\beta_c}, e^{-2\beta_c}) \in \partial D \qquad \beta_c = 0.8276\cdots$$

$$\lambda = 2x_c + 3x_c^2$$ (as in Thms 1 and 2)



Also for 3-dim Sierpinski gasket, ν and the scaling limit are known. (T.Hattori, Kusuoka, K.H. 1993)
$\rightarrow$ 4-dimensional dynamical system.

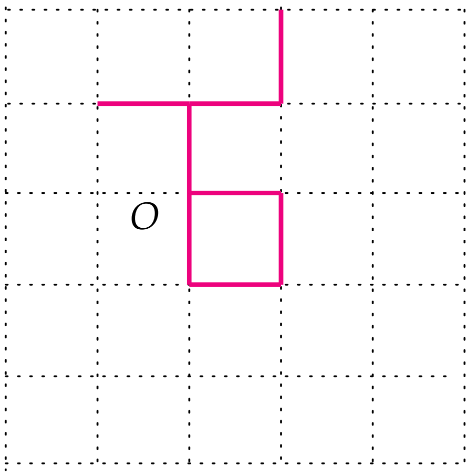


Some results for general d-dim SG's. (T.Hattori, Tsuda 2002)
m-gasket (Kasuga , master's thesis)

2. Loop-erased random walk

Simple random walk on a graph

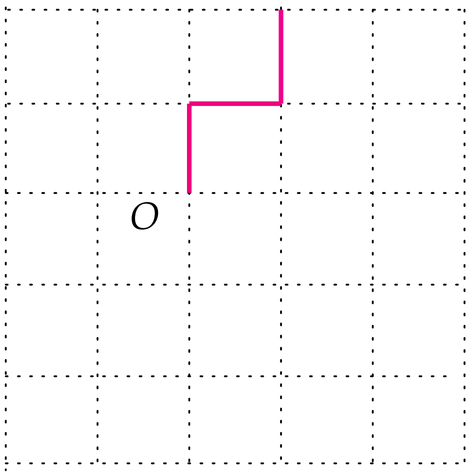
Jumps to a nearest neighbor with equal probability.



2. Loop-erased random walk

Simple random walk on a graph

Erase loops from SRW chronologically.



LERW is self-avoiding, but the distribution is different from SAW. (Lawler 1980)

Sierpinski gasket

Physicists knew (growth exponent, D.Dhar, A.Dhar (1997))

$$\nu = 1/d_{LERW} = \log 2/\log \lambda_{LERW} = 0.83\ldots$$

$$\lambda_{LERW} = (20 + \sqrt{205})/15.$$

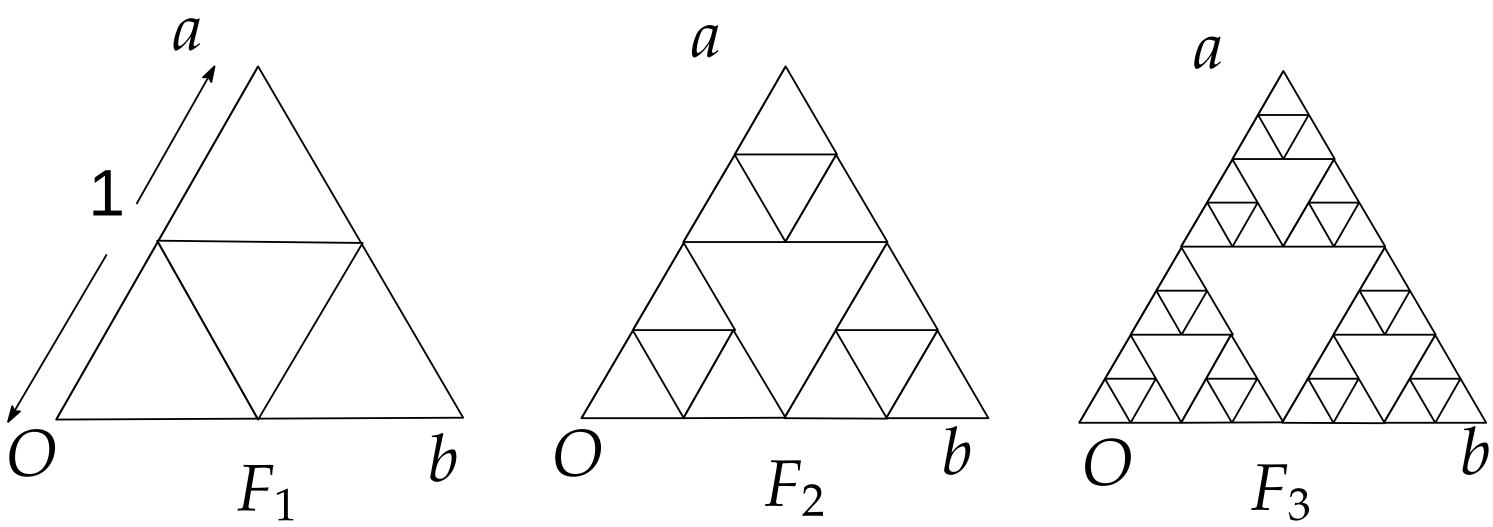
Mathematicians proved (the existence of the scaling limit and) (2014)

$$d_{LERW} = \log \lambda_{LERW}/\log 2.$$

However, $E[|w(n)|^2] \sim n^{2/d_{LERW}}$? open!

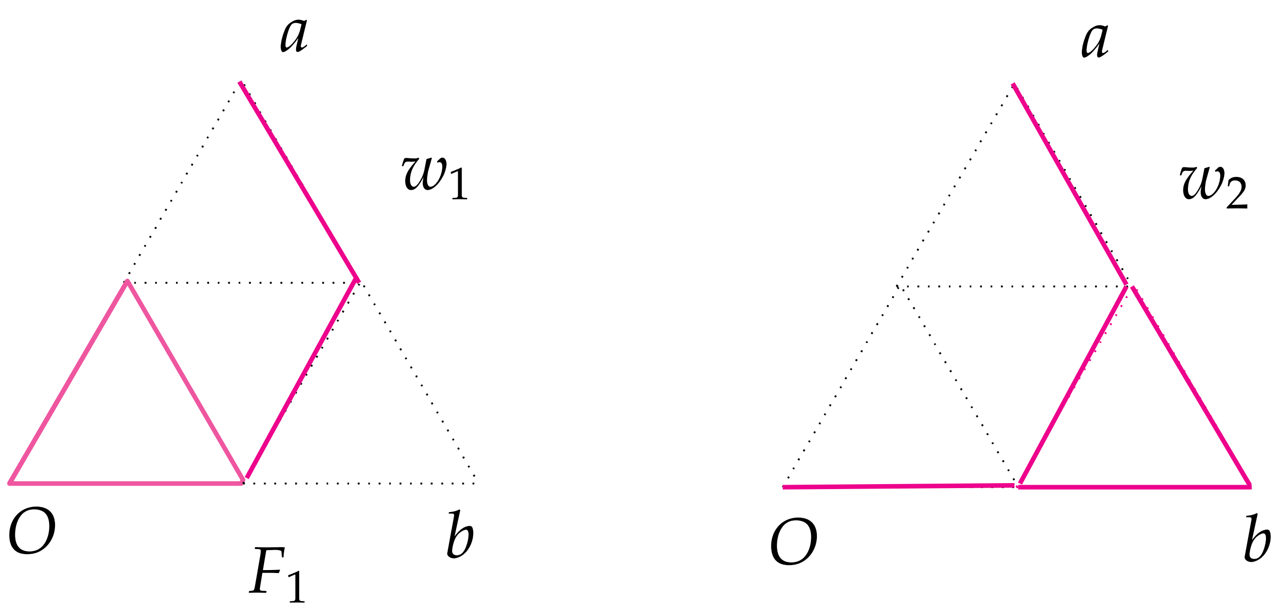
Notations

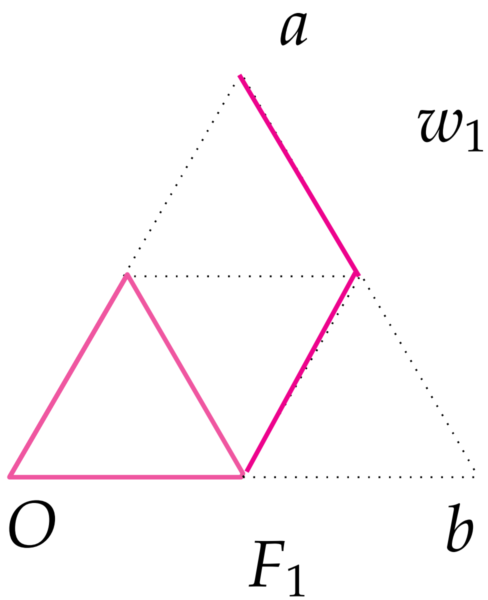
$\triangle Oab$: a unit triangle. $F_N = 2^{-N} G_N$ a graph with edge length 2^{-N}. Sierpinski gasket $F = \cup_{N=1}^{\infty} F_N$

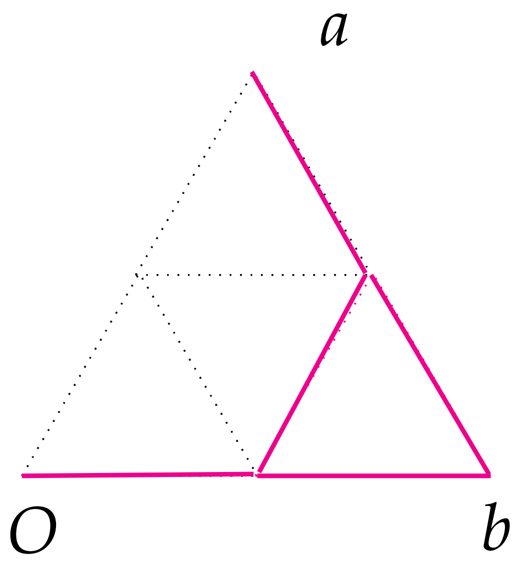


2-1. Erasing-larger-loops-first model (ELLF)

Z_N : Simple random walk on F_N, starting at O and stopped at a.







Two conditioned simple random walks on F_N from O to a.

P_N : the path measure of SRW not via b.

P'_N : the path measure of SRW via b.

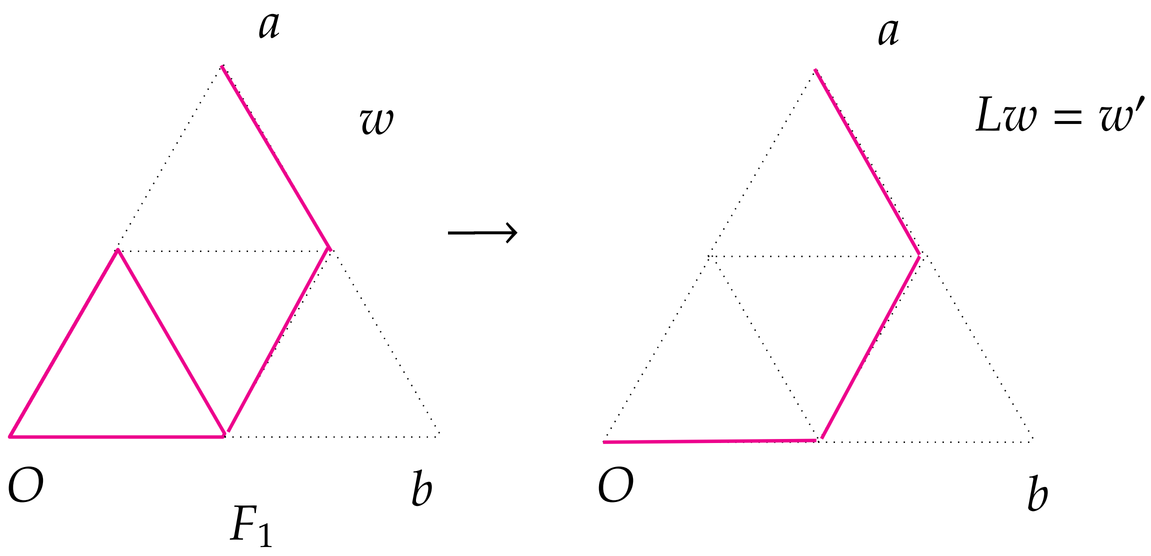
For example, (note $Z_1(0) = O, Z_1(L(w_1)) = a$)

$P_1[w_1] = P[\ Z_1(i) = w_1(i),\ i = 0, 1, 2, \cdots L(w_1)\ ]$

$= (\frac{1}{2})^2(\frac{1}{4})^4/(\frac{1}{2})$.

Conditioned

Loop erasure from random walks on F_1 (chronological).



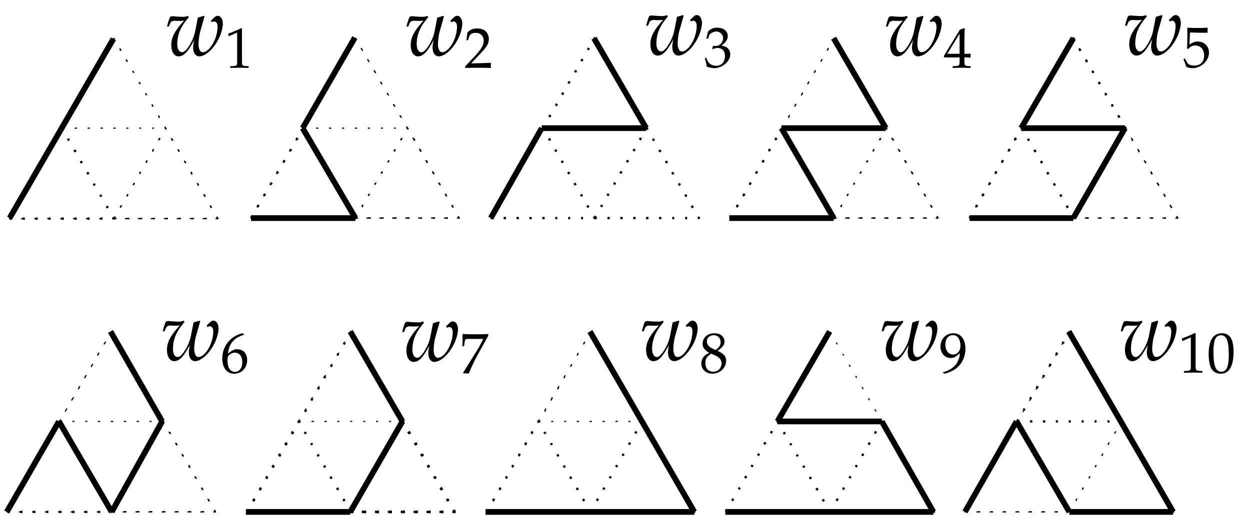
L : Loop-erasing operator.

$\hat{P}_1 = P_1 \circ L^{-1}$, $\hat{P}'_1 = P'_1 \circ L^{-1}$: LERW measures

($\hat{P}_1[w']$ is the probability to get a path w' as a result of loop-erasure.) Infinitely many paths result in a same path by L.

These probabilities can be calculated directly.

$\hat{P}_1 = P_1 \circ L_1^{-1}$: LERW measure (SRW not via b)

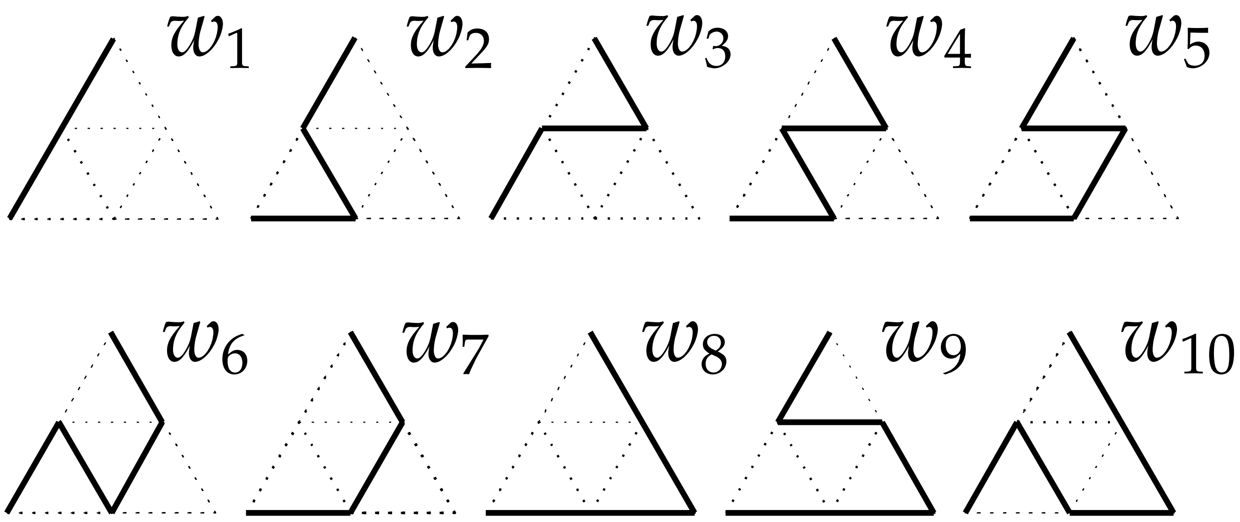


$$\hat{P}_1[w_1] = \frac{1}{2},\ \hat{P}_1[w_2] = \hat{P}_1[w_3] = \frac{2}{15},$$

$$\hat{P}_1[w_4] = \ \hat{P}_1[w_5] = \ \hat{P}_1[w_6] = \frac{1}{30},\ \hat{P}_1[w_7] = \frac{2}{15},$$

$$\hat{P}_1[w_i] = 0,\ i = 8, 9, 10.$$

$\hat{P}'_1 = P'_1 \circ L_1^{-1}$: LERW measure (SRW via b)

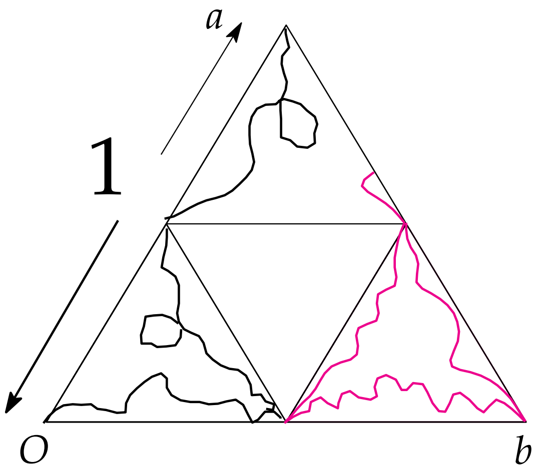


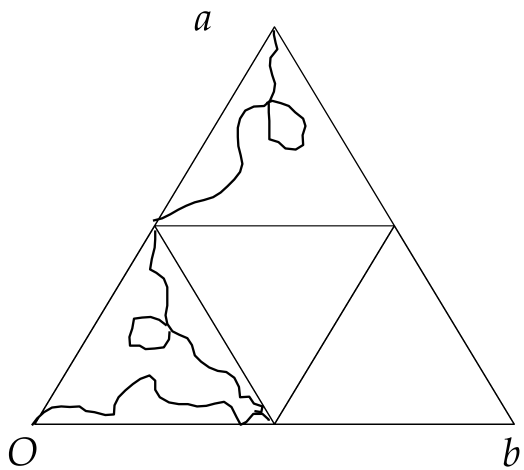
$$\hat{P}'_1[w_1] = \frac{1}{9}, \ \hat{P}'_1[w_2] = \hat{P}'_1[w_3] = \frac{11}{90},$$

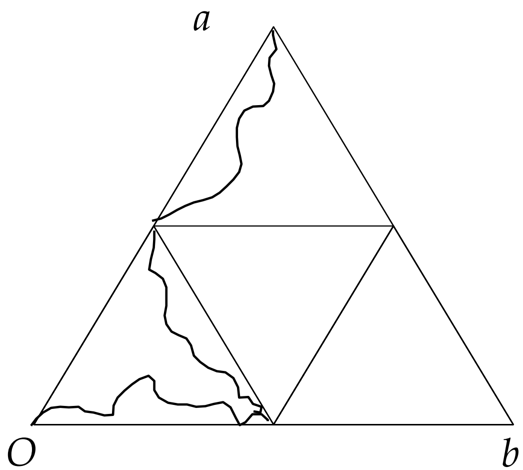
$\hat{P}'_1[w_4] = \hat{P}'_1[w_5] = \hat{P}'_1[w_6] = \frac{2}{45},$ (b can be erased)

$$\hat{P}'_1[w_7] = \frac{8}{45}, \ \hat{P}'_1[w_8] = \frac{2}{9}, \ \hat{P}'_1[w_9] = \hat{P}'_1[w_{10}] = \frac{1}{18}.$$

(A) Erasing-larger-loops-first rule (ELLF model)



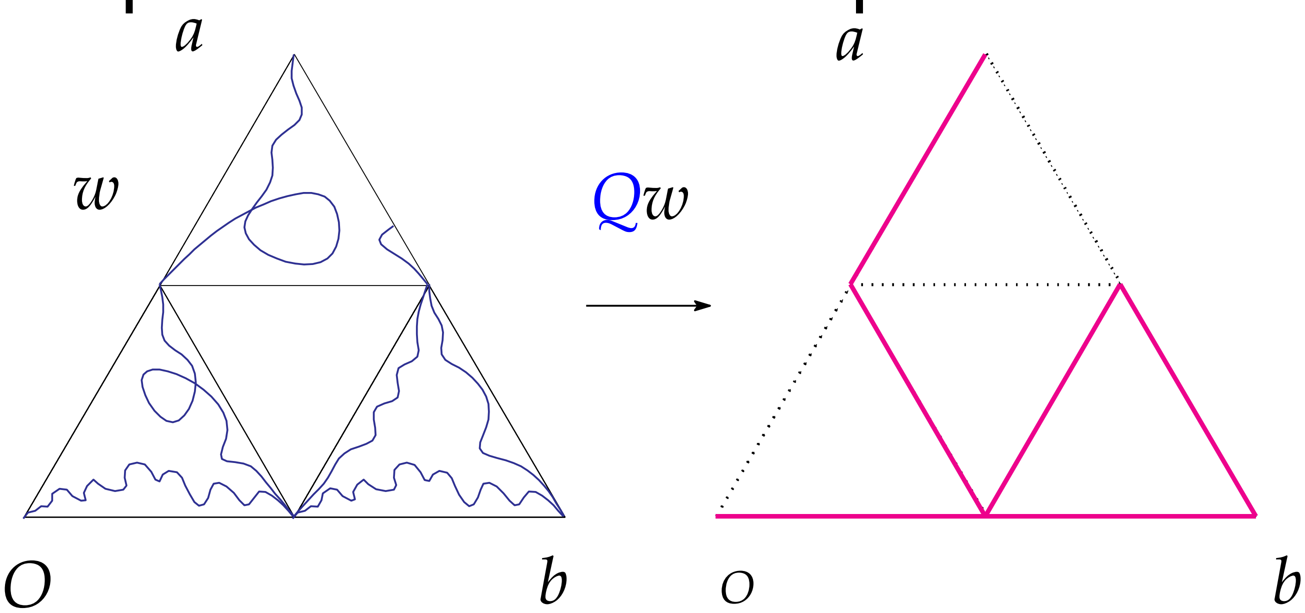




Erase loops with diameter in $(1/2, 1] \Longrightarrow$ Erase loops with diameter in $(1/4, 1/2] \Longrightarrow$ Erase loops with diameter in $(1/8, 1/4] \Longrightarrow \cdots \rightarrow$ Recursions

Erasing-larger-loops-first model (ELLF) (not chronologically)

Step 0. Step 1.

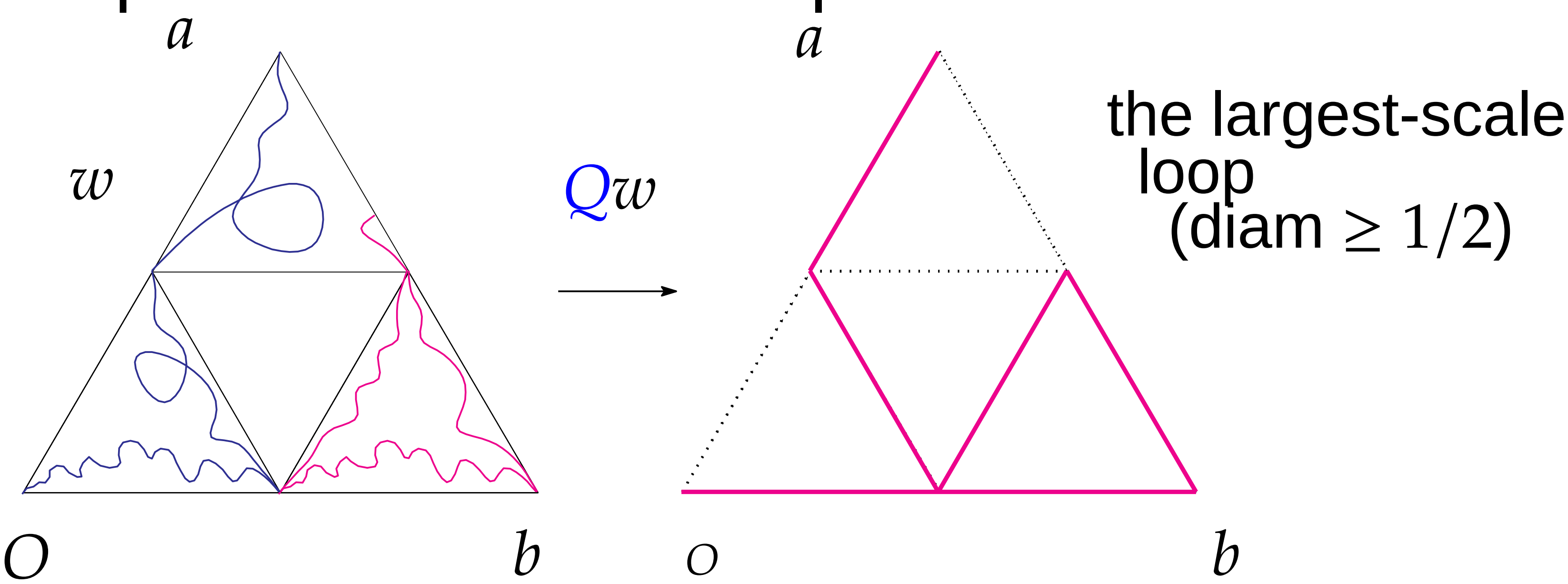


SRW on F_N
(2^{-N} - lattice)

Coarse-grained walk
(SRW on F_1)

Erasing-larger-loops-first model (ELLF) (not chronologically)

Step 0. Step 1.

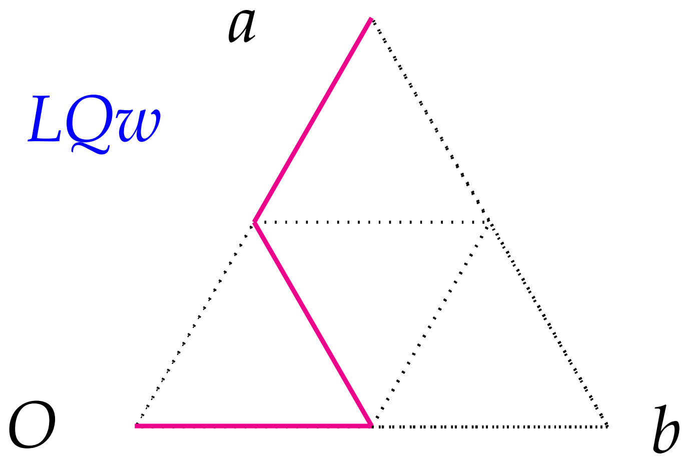


SRW on F_N (2^{-N} - lattice)

Coarse-grained walk (SRW on F_1)

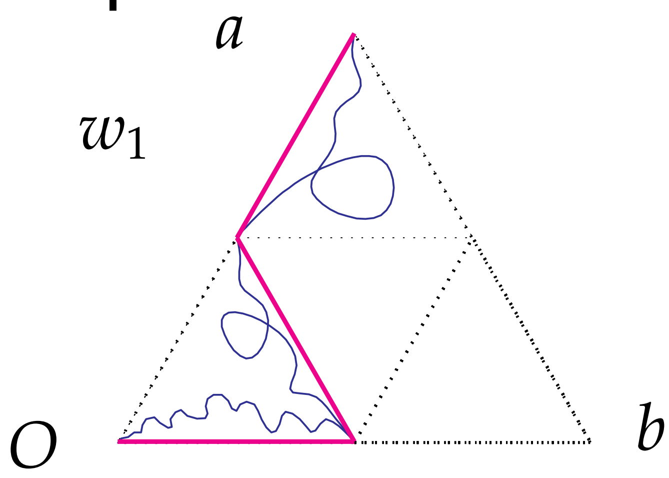
Erase loops from Qw

Step 2.

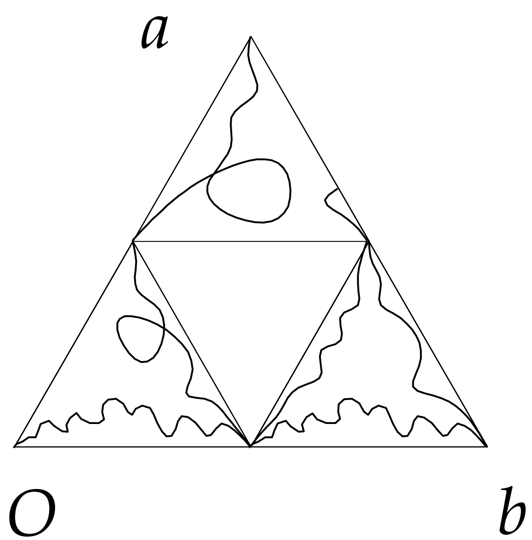


Restore fine structure

Step 3.

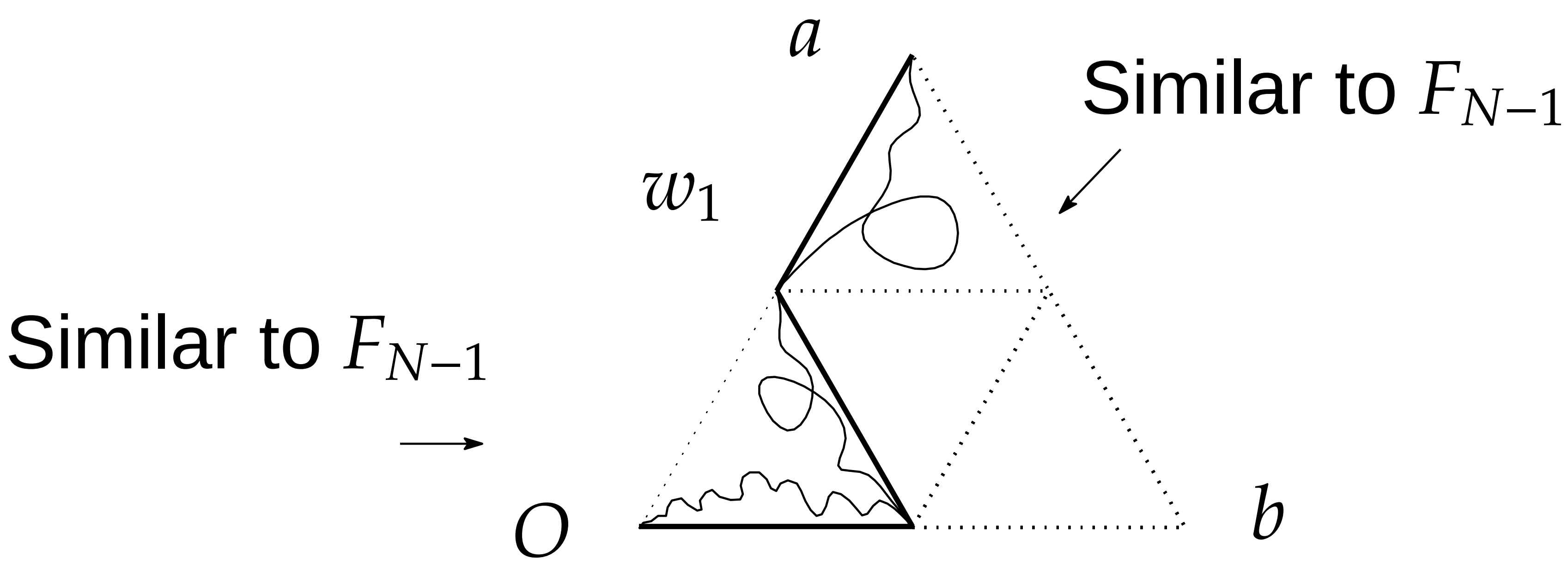


w_1 has no loops with diam $> 2^{-1}$.

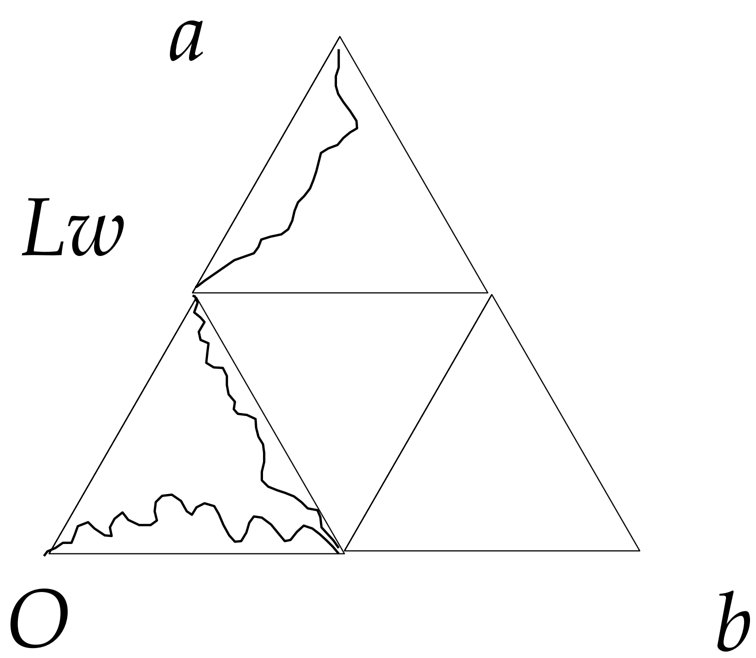


The original path

Each 2^{-1} triangle is similar to F_{N-1}. Apply Step 1–3 to each path segment and erase largest-scale (larger than $1/4$) loops. Repeat until the path has no loops.



Resulting loop-erased path (After repetition of Q and L on F_1)



L : Loop-erasing operator

$\hat{P}_N = P_N \circ L^{-1}$: LERW path meas.

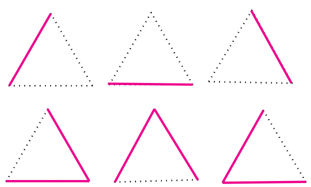
Generating functions

$\hat{W}_N$: The set of loopless paths on F_N from O to a,

$\hat{P}_N = P_N \circ L^{-1}$, $\hat{P}'_N = P'_N \circ L^{-1}$: LERW path measures

$$\Phi_N(x,y) = \sum_{w \in \hat{W}_N} \hat{P}_N(w)\, x^{s_1(w)}\, y^{s_2(w)},$$

$$\Theta_N(x,y) = \sum_{w \in \hat{W}_N} \hat{P}'_N(w)\, x^{s_1(w)}\, y^{s_2(w)}, \quad x, y \geq 0.$$



Type 1

Type 2

$s_1(w) = \sharp\{2^{-N}$-triangles of Type 1$\}$, $s_2(w) = \sharp\{$Type 2$\}$.

Recursions

$$\Phi_{N+1}(x,y) = \Phi_1(\Phi_N(x,y), \Theta_N(x,y)).$$

$$\Theta_{N+1}(x,y) = \Theta_1(\Phi_N(x,y), \Theta_N(x,y)), \quad N \in \mathbb{N}.$$

$$\Phi_1(x,y) = \frac{1}{30}(15x^2 + 8xy + y^2 + 2x^2y + 4x^3).$$

$$\Theta_1(x,y) = \frac{1}{45}(5x^2 + 11xy + 2y^2 + 14x^2y + 8x^3 + 5xy^2).$$

Mean matrix of the number of triangles

$$\mathbf{M} = \begin{bmatrix} \frac{\partial}{\partial x}\Phi_1(1,1) & \frac{\partial}{\partial y}\Phi_1(1,1) \\ \frac{\partial}{\partial x}\Theta_1(1,1) & \frac{\partial}{\partial y}\Theta_1(1,1) \end{bmatrix} = \begin{bmatrix} \frac{9}{5} & \frac{2}{5} \\ \frac{26}{15} & \frac{13}{15} \end{bmatrix}$$

The larger eigenvalue

$$\lambda_{LERW} = \frac{1}{15}(20 + \sqrt{205}) = 2.2878\ldots.$$

2-2. Answers

Thm. 3. (STW, HM, 2014)

Υ_N : LERW on F_N. $\Upsilon_N(\lambda_{LERW}^N\, t\,)$ converges uniformly in t a.s. as $N \to \infty$ to a continuous process Υ on the SG.

Thm. 4. (STW, HM 2014)

Υ is almost surely self-avoiding. (Not obvious)

The path Hausdorff dimension is

$d_{LERW}(\Upsilon([0,\infty))) = \log \lambda_{LERW} / \log 2 = 1.1939\ldots > 1$ a.s.

Thm. 5. (Mizuno, K.H. 2014) (B)

ELLF LERW $\overset{\mathrm{d}}{=}$ 'standard' LERW. (Not obvious)

Thms 3, 4 were proved by two groups independently. Shinoda, Teufl and Wagner used uniform spanning tree and obtained more detailed properties of the limit paths. Hattori, Mizuno used the erasing-larger-loops-first rule.

LERW and SAW belong to different universal classes.

$$d_{LERW} = \frac{\log(20 + \sqrt{205})/15}{\log 2} = 1.1939\ldots$$

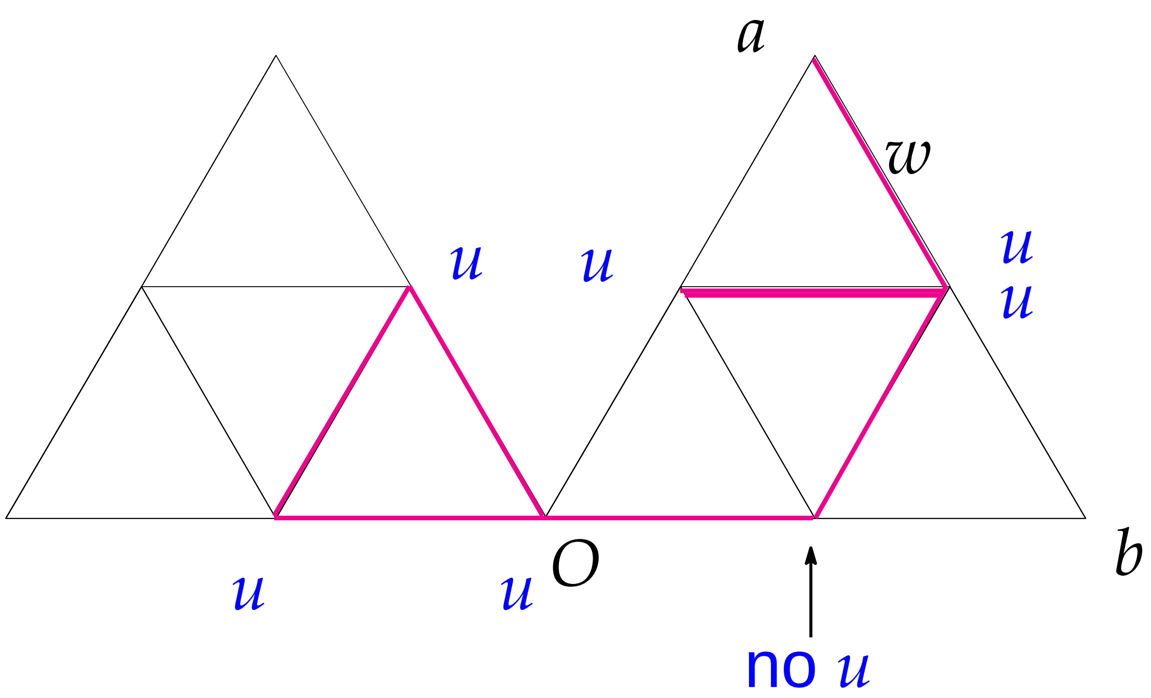
$$d_{SAW} = \frac{\log(7 - \sqrt{5})/2}{\log 2} = 1.2521\ldots$$

3. Self-repelling walks and their loop-erasure

Consider paths $O \to a$. $x > 0$

Penalty u for sharp turns and returns to O. $(0 \leq u \leq 1)$

$$P_1[w] \propto u^{N(w)} x^{L(w)}$$



$N(w) = 6,\ L(w) = 8$

We can define a one-parameter family of self-repelling walks recursively.

Thm. 6 (Hambly, T. Hattori, K.H. 2002)
The scaling limit exists and connects the Brownian motion ($u = 1$) and the self-avoiding process obtained in Thm. 2 ($u = 0$) continuously in u.

Thm. 7 (T. Hattori, K.H. 2004)

$$\forall s > 0, \quad \lim_{n\to\infty} \frac{\log E[|w(n)|^s]}{\log n} = s\nu_u.$$

$$\nu_0 = \nu_{SAW}, \nu_1 = 1/2.$$

Applying the erasing-larger-loops-first rule to this family of self-repelling walks, we obtain a new one-parameter family of walks whose paths are self-avoiding.
The scaling limit exists.
For $u = 1$, it is the limit of the LERW in Thm. 3, and for $u = 0$, the self-avoiding process in Thm. 1.

4. Summary

We considered two basic questions:

(1) Displacement exponent

(2) Scaling limit

for three kinds of non-Markov processes on the SG:

1. Self-avoiding walks (SAW)
2. Loop-erased random walks (LERW)
3. Self-repelling walks (and their loop-erasure)

Approach:

(A) Find a model that yields recursions.

(B) Extract information on ‘standard’ models.

References

- B.M. Hambly, K. Hattori, T. Hattori, *Self-repelling walk on the Sierpinski gasket*, PTRF, 124 (2002) 1-25

- M. Shinoda, E.Teufl, S. Wagner, *Uniform spanning trees on Sierpinski graphs*, arXiv:1305.5114

- K. Hattori, M. Mizuno, *Loop-erased random walk on the Sierpinski gasket*, SPA, 124 (2014) 566-585

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