A family of self-avoiding random walks interpolating the loop-erased random walk and a self-avoiding walk on the Sierpiński gasket

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ABSTRACT

We show that the 'erasing-larger-loops-first' (ELLF) method, which was first introduced for erasing loops from the simple random walk on the Sierpiński gasket, does work also for non-Markov random walks, in particular, self-repelling walks to construct a new family of self-avoiding walks on the Sierpiński gasket. The one-parameter family constructed in this method continuously connects the loop-erased random walk and a self-avoiding walk which has the same asymptotic behavior as the 'standard' self-avoiding walk. We prove the existence of the scaling limit and study some path properties: The exponent ν governing the short-time behavior of the scaling limit varies continuously in u. The limit process is almost surely self-avoiding, while it has path Hausdorff dimension $1/\nu$, which is strictly greater than 1.

Key words: loop-erased random walk ; self-avoiding walk; self-repelling walk ; scaling limit ; displacement exponent ; fractal dimension ; Sierpinski gasket ; fractal

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1 Introduction

The self-avoiding walk (SAW) and the loop-erased random walk (LERW) are two typical examples of non-Markov random walks on graphs. The self-avoiding walk is defined by the uniform measure on self-avoiding paths of a given length. In this paper we call this model the 'standard' self-avoiding walk ('standard' SAW), for we shall deal with a family of different walks whose paths are self-avoiding. The loop-erased random walk, introduced by G. Lawler ([15]), is a random walk obtained by erasing loops from the simple random walk in chronological order (as soon as each

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loop is made). Although the LERW has self-avoiding paths, it has a different distribution from that of the 'standard' SAW.

Two of the basic questions concerning random walks are:
(1) What is the asymptotic behavior of the walk as the number of steps tends to infinity? To be more specific, if $X(n)$ denotes the location of the walker starting at the origin after n steps, does the mean square displacement show a power behavior? In other words, does the following hold in some sense?

$$E[|X(n)|^2] \sim n^{2\nu},$$

where $|X(n)|$ denotes the Euclidean distance from the starting point and ν is a positive constant. If it is the case, what is the value of the displacement exponent ν?
(2) Does the walk have a scaling limit? A scaling limit is the limit as the edge length of the graph tends to 0. To give some examples, Brownian motions on $\mathbb{Z}^d$ and the Sierpiński gasket are obtained as the scaling limit of the simple random walk on the respective graph. The displacement exponent ν governs also the short-time behavior of the scaling limit.

Question (1) originated from the problem of the end-to-end distance of long polymers. Since no two monomers can occupy the same place, a self-avoiding walk is expected to model polymers. There have been many works, not only mathematical works, but also computer simulations and heuristics aimed at answering the question, however, for 'standard' self-avoiding walk on $\mathbb{Z}^d$ with $d = 2, 3, 4$, it is not solved rigorously yet. Question (2) for $\mathbb{Z}^d$, $d = 2, 3, 4$ has not been given a rigorous answer yet, either, while for $\mathbb{Z}^d$ with $d > 4$ the answers are given; the scaling limit is the d-dimensional Browinan motion and $\nu = 1/2$. The difficulties for $d = 2, 3, 4$ lie in the strong self-avoiding effect in low dimensions. For what is known about 'standard' self-avoiding walks on $\mathbb{Z}^d$, see [18].

The situation is quite different for LERW on $\mathbb{Z}^d$. The existence of the scaling limit has been proved for all d, and the asymptotic behavior has been studied in terms of the growth exponent (the reciprocal of the displacement exponent). For $d = 2$ Schramm-Loewner evolution (SLE) has played an essential role. To cite just a few about the LERW on $\mathbb{Z}^d$, see [17], [19], [13] and [16].

The Sierpiński gasket provides with a space which is 'low-dimensional', but permits rigorous analysis. For this fractal space, the displacement exponent ν of the 'standard' SAW is obtained in [10]. The scaling limit is studied in [6] and it is proved that the same ν governs the short-time behavior of the limit process X_t, that is, there exist positive constants C_1 and C_2 such that

$$C_1 \leqq \frac{E[|X_t|]}{t^\nu} \leqq C_2$$

holds for small enough t ([3]). As for the LERW, the scaling limit is obtained by two groups independently, using different methods ([20], [11]).

SLE mentioned above is a profound theory, which goes far beyond the investigation of the scaling limit of the LERW on $\mathbb{Z}^2$. It is a unified theory of a variety of random curves in $\mathbb{R}^2$ that involves a parameter κ, and different values of κ correspond to different models. $\kappa = 2$ corresponds to the scaling limit of the LERW and $\kappa = 8/3$ is conjectured to be the scaling limit of the SAW. Thus, SLE is expected to connect the SAW and the LERW on $\mathbb{R}^2$.

There arises a natural question: Is it possible to construct a model that connects the SAW and the LERW on the Sierpiński gasket continuously in some parameter? In this case we cannot use SLE, for which the conformal invariance of models in $\mathbb{R}^2$ plays an essential role.

In this paper, we construct a one-parameter family of self-avoiding random walks on the Sierpiński gasket continuously connecting the LERW and a SAW which has the same asymptotic behavior as the 'standard' SAW. We prove the existence of the scaling limit and show some path properties: The exponent ν governing the short-time behavior of the scaling limit varies continuously in u. The limit process is almost surely self-avoiding, while it has path Hausdorff dimension $1/\nu$, which is strictly greater than 1.

Main ingredients for the model are the one-parameter family of self-repelling walks on the Sierpinski gasket studied in [3] and [7], and the 'erasing-larger-loops-first' (ELLF) method employed in the study of the LERW [11]. A self-repelling walk is a walk that is discouraged, if not prohibited, to return to points it has visited before. There have been a variety of models on $\mathbb{Z}$. See, for example, the survey paper [12] and the references therein. The model we use here is unique in the way of discouraging returns; penalties are given for backtracks and sharp turns, rather than for revisits to same points or same edges.

For the 'standard' LERW on graphs, the uniform spanning tree proves to be a powerful tool ([20]). By 'standard', we mean the loops are erased chronologically as Lawler first introduced. On the other hand, [11] constructed a LERW on the Sierpiński gasket by ELLF, that is, by erasing loops in descending order of size of loops and proved that the resulting LERW has the same distribution as that of the 'standard' LERW. The uniform spanning tree is powerful in the sense that it can be used on any graphs, however, this tool is valid only for loop-erasure from *simple* random walks. We prove that ELLF does work also for other kinds of random walks on some fractals, in particular, for self-repelling walks on the Sierpiński gasket, for the method is based on self-similarity. Thus, our construction is performed by erasing loops from the family of self-repelling walks by the ELLF method.

In Section 2, we describe the set-up and recall the family of self-repelling walks introduced in [3] and [7] in a more concise manner. In Section 3, we describe the ELLF method of loop-erasing in a more organized manner than [11], and apply it to the self-repelling walks to obtain a new family of self-avoiding walks interpolating LERW and SAW. In Section 4 we study the scaling limit. In Section 5 we prove some properties of the limit process concerning the short-time behaviors. In Section 6, we give the conclusion and some remarks.

2 Self-repelling walk on the pre-Sierpiński gaskets

Let us first recall the definition of the pre-Sierpiński gaskets, that is, graph approximations of the Sierpiński gasket which is a fractal with Hausdorff dimension $\log 3/\log 2$. Let $O = (0,0)$, $a = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $b = (1,0)$ and define F'_0 to be the graph that consists of the three vertices and the three edges of $\triangle Oab$. Define similarity maps $f_i : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1,2,3$ by

$$f_1(x) = \frac{1}{2}x,\ f_2(x) = \frac{1}{2}(x+a),\ f_3(x) = \frac{1}{2}(x+b),$$

and a recursive sequence of graphs $\{F'_N\}_{N=0}^{\infty}$ by

$$F'_{N+1} = f_1(F'_N) \cup f_2(F'_N) \cup f_3(F'_N).$$

Let F_N be the union of F'_N and its reflection with respect to the y-axis, and let G_N and E_N be the sets of the vertices and of the edges of F_N, respectively. F_3 is shown in Fig. 1.

Let $\mathcal{T}_M$ be the set of all upward (closed and filled) triangles which are translations of $2^{-M}\triangle Oab$ and whose vertices are in G_M; an element of $\mathcal{T}_M$ is called a 2^{-M}**-triangle**.

For each $N \in \mathbb{Z}_+ = \{0,1,2,\ldots\}$, denote the set of finite paths on F_N starting from O and stopped at the first hit at a by

$$W_N^0 = \{\ w = (w(0), w(1), \cdots, w(n)) :\ w(0) = O,\ w(n) = a,\ w(i) \in G_N,$$
$$\{w(i), w(i+1)\} \in E_N,\ w(i) \neq a,\ 0 \leqq i \leqq n-1,\ n \in \mathbb{N}\ \}$$

and the set of paths that do not hit any vertices in G_0 other than O on the way by

$$W_N = \{\ w = (w(0), w(1), \cdots, w(n)) \in W_N^0 :\ w(i) \notin G_0 \setminus \{O\},\ 0 \leqq i \leqq n-1,\ n \in \mathbb{N}\ \}.$$

For a path $w = (w(0), w(1), \cdots, w(n)) \in W_N^0$, denote the number of steps by $\ell(w) := n$.

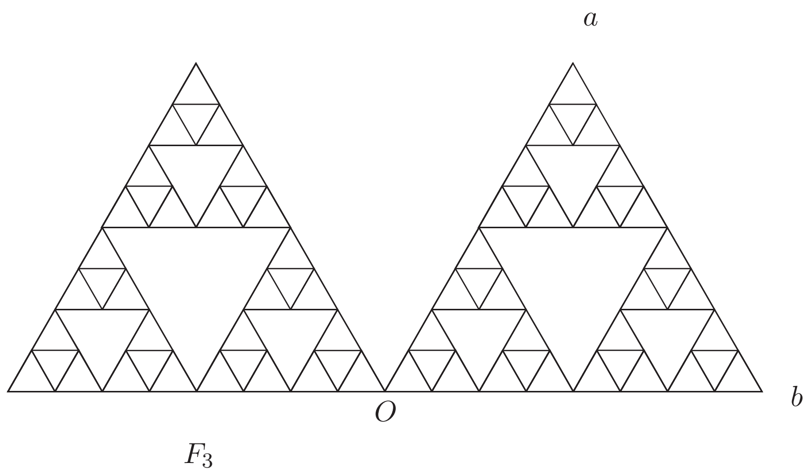

Fig 1: F_3

If we assign probability $(1/4)^{\ell(w)-1}$ to each $w \in W_N$, then we have the simple random walk on F_N starting from O and stopped at the first hit at a conditioned that the walk does not hit any vertices in $G_0 \setminus \{O\}$ on the way. $(1/4)^{-1}$ comes from this conditioning.

We shall assign probabilities such that they give random walks whose revisits to same points are discouraged. First let us start with paths in W_1. The idea is that we give a penalty to $w \in W_1$ every time it makes a sharp turn or a backtrack at $G_1 \setminus G_0$, or revisits O. We realize it by using $N(w)$, the total number of sharp turns and backtracks, and $M(w)$, the total number of revisits to O, and by assigning probability $u^{N(w)+M(w)} x_u^{\ell(w)-1}$, where u is a parameter taking values in $[0,1]$ and x_u is a positive constant determined so that the sum of the probabilities over W_1 equals to 1. This is a natural way to define a self-repelling walk on F_1: If $u = 1$, then we have $x_1 = 1/4$ and the simple random walk above, and if $u = 0$, then the probability is supported on a set of self-avoiding paths. On a general W_N, we define the probability recursively.

To give a precise definition, we shall make some preparations. For a path $w \in \bigcup_{N=1}^{\infty} W_N^0$ and $A \subset \mathbb{R}^2$, we define the hitting time of A by

$$T_A(w) = \inf\{j \geqq 0 : \ w(j) \in A\},$$

where we set $\inf \emptyset = \infty$. For $w \in W_N^0$ and $M \leqq N$, we shall define a recursive sequence $\{T_i^M(w)\}_{i=0}^m$ of **hitting times of** G_M as follows: Let $T_0^M(w) = 0$, and for $i \geqq 1$, let

$$T_i^M(w) = \inf\{j > T_{i-1}^M(w) : \ w(j) \in G_M \setminus \{w(T_{i-1}^M(w))\}\},$$

here we take m to be the smallest integer such that $T_{m+1}^M(w) = \infty$. Then $T_i^M(w)$ is the time (steps) taken for the path w to hit vertices in G_M for the $(i+1)$-th time, under the condition that if w hits the same vertex in G_M more than once in a row, we count it only 'once'.

For each $M \in \mathbb{Z}_+$, we define a coarse-graining map $Q_M : \bigcup_{N=M}^{\infty} W_N^0 \to W_M^0$ by setting $(Q_M w)(i) = w(T_i^M(w))$ for $i = 0, 1, 2, \ldots, m$, where m is as above. Note that

$$Q_K \circ Q_M = Q_K, \qquad \text{if} \quad K \leqq M$$

holds and that if $w \in W_N$ and $M \leqq N$, then $Q_M w \in W_M$.

For $w \in W_1$, define the reversing number $N(w)$ and the revisiting number $M(w)$ by

$$N(w) = \sharp\{1 \leqq i \leqq \ell(w) - 1 \ : \ \overrightarrow{w(i-1)w(i)} \cdot \overrightarrow{w(i)w(i+1)} < 0, \ w(i) \notin G_0 \ \}, \tag{2.1}$$

$$M(w) = \sharp\{1 \leqq i \leqq \ell(w) \ : \ w(i) = O\}, \tag{2.2}$$

where $\overrightarrow{a} \cdot \overrightarrow{b}$ denotes the inner product of $\overrightarrow{a}$ and $\overrightarrow{b}$ in $\mathbb{R}^2$.

For $x > 0$ and $0 \leqq u \leqq 1$, define

$$\Phi(x,u) = \sum_{w \in W_1} u^{N(w)+M(w)} \ x^{\ell(w)}. \tag{2.3}$$

For each u, within the radius of convergence $r_u > 0$ as power series in x, we have the following explicit form of Φ given in [3]:

$$\Phi(x,u) = \frac{x^2\{1+(1+u)x - u(1-u^2)x^2 + 2(1-u)^2u^2x^3\}}{(1+ux)(1-2ux) - 4u^2x^2\{1+2(1-u^2)x^2 - 2u(1-u)^2x^3\}}.$$

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Proposition 1 *(Proposition 2.3 in [3])*

(1) *For each $u \in [0,1]$, there is a unique fixed point x_u of the mapping $\Phi(\cdot,u) : (0,r_u) \to (0,\infty)$, that is, $\Phi(x_u,u) = x_u, x_u > 0$. As a function in u, x_u is continuous and strictly decreasing on $[0,1]$.*

(2) *Let $\tilde{\lambda}_u = \dfrac{\partial \Phi}{\partial x}(x_u,u)$. Then $\tilde{\lambda}_u > 2$ and $\tilde{\lambda}_u$ is continuous in u.*

In the two extreme cases, we know that $x_0 = \dfrac{\sqrt{5}-1}{2}$, $\tilde{\lambda}_0 = \dfrac{7-\sqrt{5}}{2}$, and $x_1 = \dfrac{1}{4}$, $\tilde{\lambda}_1 = 5$.

To define a family of probability measures $\{P_N^u, \ u \in [0,1]\}$ on each W_N, we consider decompositions of a path based on the self-similarity and the symmetries of the pre-Sierpiński gaskets. Assume $w \in W_N$ and $0 \leqq M < N$ and denote $\tilde{w} = Q_M w$. Since the pair of adjacent 2^{-M}-triangles including $\tilde{w}(i-1)$, $\tilde{w}(i)$ and $\tilde{w}(i+1)$ is similar to F_{N-M}, there is a unique decomposition

$$(\tilde{w}; w_1, \cdots, w_{L(\tilde{w})}), \ \tilde{w} \in W_M, \ w_i \in W_{N-M}, \ i = 1, \cdots, \ell(\tilde{w}) \tag{2.4}$$

such that the path segment $(w(T_{i-1}^M(w)), w(T_{i-1}^M(w)+1)), \cdots, w(T_i^M(w)))$ of w is identified with $w_i \in W_{N-M}$ by appropriate similarity, rotation, translation and reflection so that $w(T_{i-1}^M(w))$ is identified with O and $w(T_i^M(w))$ with a. We shall use this kind of identification throughout this paper. We illustrate a simple example of the decomposition in Fig. 2.

First, for each $w \in W_1$, let

$$P_1^u(w) = u^{N(w)+M(w)} \ x_u^{\ell(w)-1}, \tag{2.5}$$

and define P_N^u on W_N recursively by

$$P_N^u(w) = P_{N-1}^u(\tilde{w}) \prod_{i=1}^{L(\tilde{w})} P_1^u(w_i), \tag{2.6}$$

where $(\tilde{w}; w_1, \cdots, w_{L(\tilde{w})})$ is the decomposition of $w \in W_N$ with $M = N-1$ given in (2.4). Denote the image measure of P_N^u induced by the mapping Q_M by $P_N^u \circ Q_M^{-1}$. P_N^u is self-similar in the sense that $P_N^u \circ Q_M^{-1} = P_M^u$.

$(W_N, \{P_N^u\}_{u \in [0,1]})$ defines a family of self-repelling walks Z_N^u on F_N such that

$$Z_N^u(w)(i) = w(i), \quad i = 0, \cdots, \ell(w), \quad w \in W_N. \tag{2.7}$$

In [3], it is proved that for each u, the sequence $\{Z_N^u(\tilde{\lambda}_u^N \ \cdot \ )\}_{N=1}^{\infty}$ of time-scaled self-repelling walks converges to a continuous process as $N \to \infty$. The one-parameter family of the limit

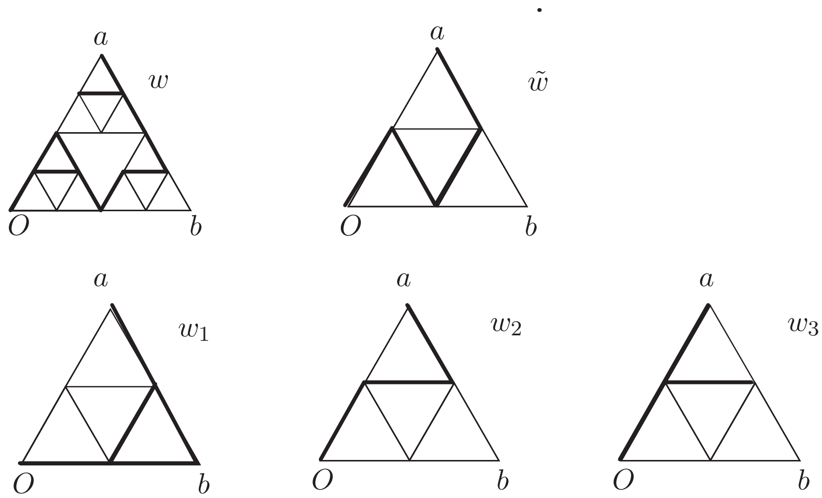

Fig 2: $w, \tilde{w}, w_1, w_2, w_3$

processes $\{Z^u(\ \cdot\ ), u \in [0,1]\}$ continuously interpolates a self-avoiding process ($u = 0$) and the Brownian motion ($u = 1$) on the Sierpiński gasket.

In the next section, we erase loops from this family of self-repelling walks to obtain a one-parameter family of self-avoiding walks. For this purpose, we introduce another family of self-repelling walks. Let

$$V_N = \{\ w \in W_N^0\ :\ Q_0 w = (O, b, a)\}.$$

$w \in V_N$ consists of two parts, $(w(0), w(1), \cdots, w(T_1^0(w)))$ and $(w(T_1^0(w)), w(T_1^0(w)+1), \cdots, w(T_2^0(w)))$, and they can be identified with some $w, w'' \in W_N$, respectively. Define a probability measure $P_N'^u$ on V_N by

$$P_N'^u[w] = P_N^u[w'] \cdot P_N^u[w''],$$

where P_N^u is defined in (2.5) and (2.6). $(V_N, \{P_N'^u\}_{u\in[0,1]})$ defines another family of self-repelling random walks $Z_N'^u$ on F_N such that

$$Z_N'^u(w)(i) = w(i), i = 0, \cdots, \ell(w), \quad w \in V_N. \tag{2.8}$$

This is a family of self-repelling walks that hit b 'once' in the sense that $Q_0 w = (O, b, a)$.

3 Loop-erasure by erasing-larger-scale-loops-first rule

For $(w(0), w(1), \cdots, w(n)) \in W_N^0$, if there are $c \in G_N$, i and j, $0 \leqq i < j \leqq n$ such that $w(i) = w(j) = c$ and $w(k) \neq c$ for any $i < k < j$, we call the path segment $[w(i), w(i+1), \ldots, w(j)]$ a **loop formed at c** and define its **diameter** by $d = \sup_{i \leqq k_1 < k_2 \leqq j} |w(k_1) - w(k_2)|$, where $|\ \cdot\ |$ denotes the Euclidean distance. Note that a loop can be a part of another larger loop formed at some other vertex. $W_N \cup V_N$ is the set of paths in W_N^0 that do not have any loops with diameter greater than or equal to 1. Let Γ_N be the set of loopless paths on F_N from O to a:

$$\Gamma_0 = \{(O, a), (O, b, a)\},$$

$$\Gamma_N = \{\ (w(0), w(1), \cdots, w(n)) \in W_N \cup V_N :\ w(i) \neq w(j),\ 0 \leqq i < j \leqq n,\ n \in \mathbb{N}\ \}.$$

Loop-erasure on $\mathbf{F_1}$

We shall now describe the loop-erasing procedure for paths in $W_1 \cup V_1$:

(i) Erase all the loops formed at O;

(ii) Progress one step forward along the path, and erase all the loops at the new position;

(iii) Iterate this process, taking another step forward along the path and erasing the loops there, until reaching a.

To be precise, for $w \in W_1 \cup V_1$, define the recursive sequence $\{s_i\}_{i=0}^{n}$

$$s_0 = \sup\{j : w(j) = O\},$$

$$s_i = \sup\{j : w(j) = w(s_{i-1}+1)\}.$$

If $s_i > s_{i-1}+1$, then $[w(s_{i-1}+1), w(s_{i-1}+2), \dots, w(s_i-1), w(s_i)]$ forms a loop or multiple loops at $w(s_{i-1}+1) = w(s_i)$, so we erase this part by removing $w(s_{i-1}+1), w(s_{i-1}+2), \dots, w(s_i-2)$, and $w(s_i-1)$. If $w(s_n) = a$, then we have obtained a loop-erased path,

$$Lw = [w(s_0), w(s_1), \dots, w(s_n)] \in \Gamma_1,$$

where $L : W_1 \cup V_1 \to \Gamma_1$ is the loop-erasing operator. $\square$

Fig. 3 shows all the possible loopless paths from O to a on F_1. Here only the parts in $\triangle Oab$ are shown, for any path cannot go into the other triangle without making a loop.

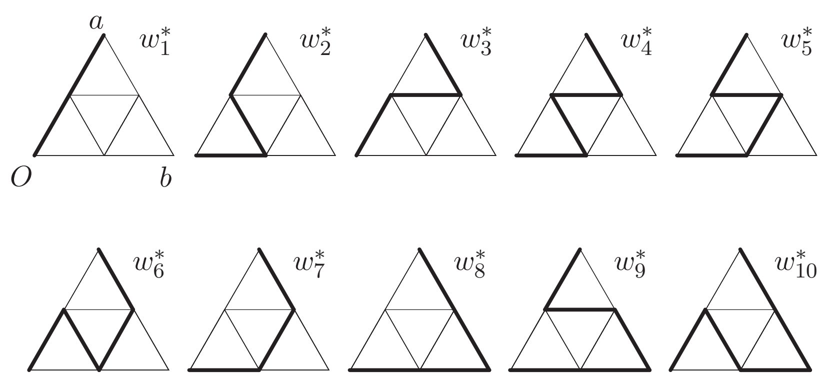

Fig 3: Loopless paths from 0 to a on F_1

Note that $w \in W_1$ implies $Lw \in W_1 \cap \Gamma_1$, but that $w \in V_1$ can result in $Lw \in W_1 \cap \Gamma_1$, with b being erased together with a loop. So far, our loop-erasing procedure is the same as the chronological method defined for paths on $\mathbb{Z}^d$ in [15].

For a general N, we erase loops from the largest scale loops down, repeatedly applying the loop-erasing procedure on F_1.

First step of the induction – erasing largest scale loops

We shall illustrate the first step of loop-erasure. Decompose a path $w \in W_N \cup V_N$ into $(Q_1 w; w_1, \cdots, w_{\ell(Q_1 w)})$, $w_i \in W_{N-1} \cup V_{N-1}$ $i = 1, \cdots, \ell(Q_1 w)$ as in (2.4). Fig. 4(a) shows $w \in W_N \cup V_N$ and Fig. 4(b) shows $Q_1 w$. Erase all the loops in chronological order from $Q_1 w \in W_1 \cup V_1$ to obtain $LQ_1 w$ as in Fig. 4(c), then restore the original fine structures to the remaining parts as shown in Fig. 4(d). That is, if we write

$$LQ_1 w = [w(T_0^1), w(T_{s_1}^1), \dots, w(T_{s_n}^1)],\ n = \ell(LQ_1 w),$$

for each i, fit the path segment $w_{s_i+1} = (w(T_{s_i}^1), w(T_{s_i}^1 + 1), \cdots, w(T_{s_i+1}^1))$ between $w(T_{s_i}^1)$ and $w(T_{s_{i+1}}^1)$ of $LQ_1 w$. We call the path obtained at this stage $\tilde{L}w$. Notice that in this stage all the

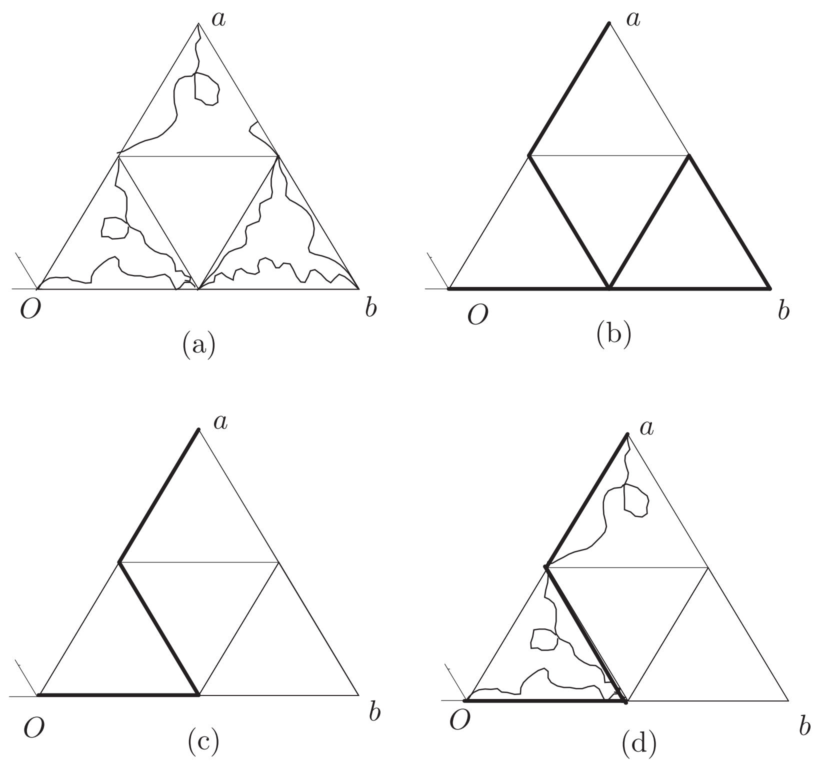

Fig 4: The loop-erasing procedure: (a) w, (b) $Q_1 w$, (c) $LQ_1 w$, (d) $\tilde{L}w$

loops with diameter greater than 1/2 have been erased. Let $\hat{Q}_1 w = LQ_1 w$. This completes the first induction step. $\square$

The idea is to repeat a similar procedure within each 2^{-1}-triangle to erase all loops with diameter greater than 1/4, and then within each 4^{-1}-triangle, and so on, until there remain no loops. To describe next induction steps more precisely, we make some preparations. For $w \in W_N^0$ and $M \leqq N$, we shall define the sequence $(\Delta_1, \dots, \Delta_k)$ of the 2^{-M}-triangles w 'passes through', and their exit times $\{T_i^{ex,M}(w)\}_{i=1}^k$ as a subsequence of $\{T_i^M(w)\}_{i=1}^m$ as follows: Let $T_0^{ex,M}(w) = 0$. There is a unique element of $\mathcal{T}_M$ that contains $w(T_0^M)$ and $w(T_1^M)$, which we denote by Δ_1. For $i \geqq 1$, define

$$J(i) = \min\{j \geqq 0 \ : \ j < m, \ T_j^M(w) > T_{i-1}^{ex,M}(w), \ w(T_{j+1}^M(w)) \notin \Delta_i\},$$

if the minimum exists, otherwise $J(i) = m$. Then define $T_i^{ex,M}(w) = T_{J(i)}^M(w)$, and let Δ_{i+1} be the unique 2^{-M}-triangle that contains both $w(T_i^{ex,M})$ and $w(T_{J(i)+1}^M)$. By definition, we see that $\Delta_i \cap \Delta_{i+1}$ is a one-point set $\{w(T_i^{ex,M})\}$, for $i = 1, \dots, k-1$. We denote the sequence of these triangles by $\sigma_M(w) = (\Delta_1, \dots, \Delta_k)$, and call it the **$\mathbf{2^{-M}}$-skeleton** of w. We call the sequence $\{T_i^{ex,M}(w)\}_{i=0,1,\dots,k}$ **exit times** from the triangles in the skeleton. For each i, there is an $n = n(i)$ such that $T_{i-1}^{ex,M}(w) = T_n^M(w)$. If $T_i^{ex,M}(w) = T_{n+1}^M$, we say that $\Delta_i \in \sigma_M(w)$ is **Type 1**, and if $T_i^{ex,M}(w) = T_{n+2}^M$, **Type 2**. If $w \in \Gamma_N$ and $M \leqq N$, then its 2^{-M}-skeleton is a collection of distinct 2^{-M}-triangles and each of them is either Type 1 or Type 2. Assume $w \in W_N \cup V_N$ and $M \leqq N$. For each Δ in $\sigma_M(w)$, the **path segment of w in $\mathbf{\Delta}$** is defined by

$$w|_\Delta = [w(n), \ T_{i-1}^{ex,M}(w) \leqq n \leqq T_i^{ex,M}]. \tag{3.1}$$

Note that the definition of T_i^M allows a path segment $w|_\Delta$ to leak into two neighboring 2^{-M}-triangles.

If $Q_M w \in \Gamma_M$, then $w|_\Delta \in W_{N-M}$ or $w|_\Delta \in V_{N-M}$, according to the type of $\Delta \in \sigma_M(w)$, where the entrance to Δ is identified with O and the exit with a. This means that each w satisfying $Q_M w \in \Gamma_M$ can be decomposed uniquely to

$$(\sigma_M(w);\ w|_{\Delta_1}, \cdots, w|_{\Delta_k}),\ w|_{\Delta_i} \in W_{N-M} \cup V_{N-M},\ i = 1, \cdots, k. \tag{3.2}$$

Conversely, given a collection of distinct 2^{-M}-triangles $\{\Delta_i\}_{i=1}^k$ such that $O \in \Delta_1$, $a \in \Delta_k$, Δ_i and Δ_{i+1} are neighbors, and $w_i' \in W_{N-M} \cup V_{N-M}$, $i = 1, \cdots, k$, then we can assemble them to obtain a unique element w of $W_N \cup V_N$ with $Q_M w \in \Gamma_M$.

We call a loop $[w(i), w(i+1), \cdots, w(i+i_0)]$ a $\mathbf{2^{-M}}$**-scale loop** whenever there exists an $M \in \mathbb{Z}_+$ such that

$$\min\{N' : w(i) = w(i+i_0) \in G_{N'}\} = M,\ d \geqq 2^{-M},$$

where d is the diameter of the loop.

Using above as a base step, we shall now describe the induction step of our operation:

Induction step

Let $w \in W_N \cup V_N$. For $1 \leqq M \leqq N$, assume that all of the 2^{-1} to 2^{-M}-scale loops have been erased from w, and denote by $w' \in W_N \cup V_N$ the path obtained at this stage. Then $Q_M w' \in \Gamma_M$.

1) Since $Q_M w' \in \Gamma_M$, we have the decomposition of w': $(\sigma_M(w');\ w_1', \cdots w_k')$, $w_i' \in W_{N-M} \cup V_{N-M}$ as given in (3.2).

2) From each w_i', erase 2^{-1}-scale loops (largest scale loops) according to the base step procedure above to obtain $\tilde{L}w_i' \in W_{N-M} \cup V_{N-M}$ and $\hat{Q}_1 w_i \in \Gamma_1$.

3) Assemble $(\sigma_M(w');\ \tilde{L}w_1', \cdots, \tilde{L}w_k')$ and $(\sigma_M(w');\ \hat{Q}_1 w_1', \cdots, \hat{Q}_1 w_k')$ to obtain $w'' \in W_N \cup V_N$ and $\hat{Q}_{M+1} w \in \Gamma_{M+1}$, respectively. w'' has no 2^{-1} to $2^{-(M+1)}$-scale loops.

□

We then continue this operation until we have erased all of the loops to have $Lw = \hat{Q}_N w \in \Gamma_N$. In this way, the loop erasing operator L defined for $W_1 \cup V_1$ has been extended to $L : \bigcup_{N=1}^\infty (W_N \cup V_N) \to \bigcup_{N=1}^\infty \Gamma_N$ with $L(W_N \cup V_N) = \Gamma_N$. Notice that the operation described above is essentially a repetition of loop-erasing for $W_1 \cup V_1$. $\hat{Q}_M$ is a map from $\bigcup_{N=M}^\infty (W_N \cup V_N)$ to Γ_M. In the induction step, we observe that $\hat{Q}_{M+1} w = Q_{M+1} w''$. Although it may occur that $\sigma_{M+1}(w'') \neq \sigma_{M+1}(w')$ because of the erasure of $2^{-(M+1)}$-scale loops, it holds that $\sigma_M(w'') = \sigma_M(w')$, which can be extended to $\sigma_K(w') = \sigma_K(w'')$ for any $K \leqq M$. We remark that the procedure implies that for any $w \in W_N \cup V_N$,

$$\sigma_K(\hat{Q}_M w) = \sigma_K(\hat{Q}_K w) \text{ for any } N \geqq M \geqq K. \tag{3.3}$$

In particular,

$$\sigma_K(Lw) = \sigma_K(\hat{Q}_K w) \text{ for } K \leqq N. \tag{3.4}$$

i.e., in the process of loop-erasing, once loops of 2^{-K}-scale and greater have been erased, the 2^{-K}-skeleton does not change any more. However, it should be noted that the types of the triangles can change from Type 2 to Type 1.

We induce measures $\hat{P}_N^u = P_N^u \circ L^{-1}$ and $\hat{P}_N'^u = P_N'^u \circ L^{-1}$, which satisfy $\hat{P}_N^u[\Gamma_N] = 1$ and $\hat{P}_N'^u[\Gamma_N] = 1$. For $w_1^*, \cdots, w_{10}^*$ shown in Fig. 3, denote

$$p_i = \hat{P}_1^u[w_i^*] = P_1^u[w : Lw = w_i^*],\quad q_i = \hat{P}_1'^u[w_i^*] = P_1'^u[w : Lw = w_i^*]. \tag{3.5}$$

p_i and q_i can be obtained as explicit functions of u and x_u by direct, but lengthy calculations, which are shown in Appendix. In the case that $u = 1$ (the ordinary loop-erased random walk), we have $x_1 = 1/4$, $p_1 = 1/2$, $p_2 = p_3 = p_7 = 2/15$, $p_4 = p_5 = p_6 = 1/30$, $q_1 = 1/9$, $q_2 = q_3 = 11/90$, $q_4 = q_5 = q_6 = 2/45$, $q_7 = 8/45$, $q_8 = 2/9$ and $q_9 = q_{10} = 1/18$ as in [11]. For $u = 0$, we have $p_1 = x_0$, $p_7 = x_0^2$ and $p_i = 0$ otherwise, and $q_1 = x_0^4$, $q_2 = q_3 = x_0^3$, $q_8 = x_0^2$ and $q_i = 0$ otherwise, with $x_0 = (\sqrt{5} - 1)/2$ as in [8].

$\hat{P}_N^u$ and $\hat{P}_N'^u$ define two families of walks obtained by erasing loops from Z_N^u and $Z_N'^u$, respectively. We remark that that $\dfrac{2}{3}\hat{P}_N^1 + \dfrac{1}{3}\hat{P}_N'^1$ equals to the 'standard' LERW studied in [20].

An important observation is that in the process of erasing loops from Z_{N+1}^u, if we stop at the point where we have obtained $\hat{Q}_N Z_{N+1}^u$, it is nothing but the procedure for obtaining LZ_N^u from Z_N^u. The same holds also for $Z_{N+1}'^u$. This can be expressed as:

$$P_{N+1}^u[\{v' : \hat{Q}_N v' = v\}] = \hat{P}_N^u[v], \quad P_{N+1}'^u[\{v' : \hat{Q}_N v' = v\}] = \hat{P}_N'^u[v]. \tag{3.6}$$

In this stage what is left to do for obtaining LZ_{N+1}^u from $\hat{Q}_N Z_{N+1}^u$ is a sequence of loop-erasing from Z_1^u or $Z_1'^u$. This combined with (3.6) leads to a 'decomposition' of LERW measures. For $w \in \Gamma_{N+1}$,

$$\begin{aligned}
\hat{P}_{N+1}^u[\ w\ ] &= \sum_{v \in \Gamma_N} P_{N+1}^u[\{v' : Lv' = w\} |\ \hat{Q}_N v' = v\ ]\ P_{N+1}^u[\{v' : \hat{Q}_N v' = v\}] \\
&= \sum_{v \in \Gamma_N} (\prod_{i=1}^{k} \hat{P}_1^{*u}[\ w_i\ ])\ \hat{P}_N^u[\ v\ ],
\end{aligned}$$

where $\sigma_N(v) = (\Delta_1, \cdots, \Delta_k)$, $w_i = v|_{\Delta_i}$ (identification implied), $\hat{P}_1^{*u} = \hat{P}_1^u$ if Δ_i is Type 1, and $\hat{P}_1^{*u} = \hat{P}_1'^u$ if Δ_i is Type 2. A similar decomposition holds also for $\hat{P}_{N+1}'^u$. This is the key to the recursion relations of generating functions defined below.

For $w \in \Gamma_N$, let us denote the number of 2^{-N}- triangles of Type 1, (the path passes two of the edges) and those of Type 2 (the path passes all three edges) in $\sigma_N(w)$ by $s_1(w)$ and $s_2(w)$, respectively. Note that $\ell(w) = s_1(w) + 2s_2(w)$. Define two sequences, $\{\hat{\Phi}_N\}_{N \in \mathbb{N}}$ and $\{\hat{\Theta}_N\}_{N \in \mathbb{N}}$, of generating functions by:

$$\hat{\Phi}_N(x, y) = \sum_{w \in \Gamma_N} \hat{P}_N^u(w) x^{s_1(w)} y^{s_2(w)},$$

$$\hat{\Theta}_N(x, y) = \sum_{w \in \Gamma_N} \hat{P}_N'^u(w) x^{s_1(w)} y^{s_2(w)}, \quad x, y \geqq 0.$$

For simplicity, we shall denote $\hat{\Phi}_1(x, y)$ and $\hat{\Theta}_1(x, y)$ by $\hat{\Phi}(x, y)$ and $\hat{\Theta}(x, y)$ and omit writing u-dependence explicitly. Similar to Proposition 3 in [11], we have

Proposition 2 *The above generating functions satisfy the following recursion relations for all* $N \in \mathbb{N}$ *:*

$$\hat{\Phi}(x, y) = p_1 x^2 + (p_2 + p_3)xy + p_4 y^2 + (p_5 + p_6)x^2 y + p_7 x^3,$$
$$\hat{\Theta}(x, y) = q_1 x^2 + (q_2 + q_3)xy + q_4 y^2 + (q_5 + q_6)x^2 y + q_7 x^3 + q_8 x^2 y + (q_9 + q_{10})xy^2,$$

$$\hat{\Phi}_{N+1}(x, y) = \hat{\Phi}_N(\hat{\Phi}(x, y), \hat{\Theta}(x, y)),$$

$$\hat{\Theta}_{N+1}(x, y) = \hat{\Theta}_N(\hat{\Phi}(x, y), \hat{\Theta}(x, y)),$$

where $p_i = \hat{P}_1^u[w_i^*]$ *and* $q_i = \hat{P}_1'^u[w_i^*]$, $i = 1, 2, \cdots, 10$.

Define the mean matrix by

$$\mathbf{M} = \begin{bmatrix} \frac{\partial}{\partial x}\Phi(1,1) & \frac{\partial}{\partial y}\Phi(1,1) \\ \frac{\partial}{\partial x}\Theta(1,1) & \frac{\partial}{\partial y}\Theta(1,1) \end{bmatrix}. \tag{3.7}$$

It is a strictly positive matrix, and the larger eigenvalue $\lambda = \lambda(u)$ is a continuous function of u, satisfying $2 < \lambda < 3$.

Let Z_N^u and $Z_N'^u$ be as in (2.7) and (2.8). The loop-erasing procedure together with the structure of the Sierpiński gasket leads to (Proposition 4 in [11])

Proposition 3 *Let $M \leqq N$. Conditioned on $\sigma_M(LZ_N^u) = (\Delta_1, \dots, \Delta_k)$ and the type of each element of the skeleton, the traverse times of the triangles*

$$T_i^{ex,M}(LZ_N^u) - T_{i-1}^{ex,M}(LZ_N^u), \quad i = 1, 2, \dots, k$$

are independent. Each of them has the same distribution as either $T_1^{ex,N-M}(LZ_{N-M}^u)$ or $T_1^{ex,N-M}(LZ'^u_{N-M})$, according to whether Δ_i is of Type 1 or Type 2.

4 The scaling limit

In this section, we investigate the limit of the loop-erased self-repelling walks constructed in Section 3 as the edge length tends to 0. Since it is easier to deal with continuous functions from the beginning, we regard F_N's as closed subsets of $\mathbb{R}^2$ made up of all the points on their edges. We define the **Sierpiński gasket** by $F = cl(\cup_{N=0}^{\infty} F^N)$, where cl denotes closure. Let

$$C = \{w \in C([0,\infty) \to F) \;:\; w(0) = O, \; \lim_{t\to\infty} w(t) = a\}.$$

C is a complete separable metric space with the metric

$$d(u,v) = \sup_{t\in[0,\infty)} |u(t) - v(t)|, \; u, v \in C,$$

where $|x - y|$, $x, y \in \mathbb{R}^2$, denotes the Euclidean distance. Hereafter, for $w \in \bigcup_{N=1}^{\infty} W_N^0$, we define

$$w(t) = a, \quad t \geqq \ell(w),$$

and interpolate the path linearly,

$$w(t) = (i + 1 - t)w(i) + (t - i)w(i + 1), \quad i \leqq t < i + 1, \quad i \in \mathbb{Z}_+$$

so that we can regard w as a continuous function on $[0, \infty)$. We shall regard also W_N, V_N and Γ_N as subsets of C. Hitting times, $\{T_i^M(w)\}_{i=1}^m$ are defined for $w \in C$ as in the previous sections, although infimum is taken over continuous time:

$$T_0^M(w) = 0, \quad T_i^M(w) = \inf\{t > T_{i-1}^M(w) \;:\; w(t) \in G_M \setminus \{w(T_{i-1}^M(w))\}\}.$$

Notice that the condition $\lim_{t\to\infty} w(t) = a$ makes $\{T_i^M(w)\}_{i=0}^m$ a finite sequence.

For $N \in \mathbb{Z}_+$, we define a coarse-graining map $Q_N : C \to C$ by $(Q_N w)(i) = w(T_i^N(w))$ for $i = 0, 1, 2, \dots, m$, and by using linear interpolation

$$(Q_N w)(t) = \begin{cases} (i+1-t)\,(Q_N w)(i) & +(t-i)\,(Q_N w)(i+1), \\ & i \leqq t < i+1, \; i = 0, 1, 2, \dots, m-1, \\ a, & t \geqq m. \end{cases}$$

We define also the 2^{-M}-**skeleton**, $\sigma_M(w)$ (a sequence of 2^{-M}-triangles w passes through), the **exit times** $\{T_i^{ex,M}\}_{i=1}^k$ and types of triangles in a similar way to their counterparts in Section 3. The loop-erasing operator is regarded as $L : \bigcup_{N=1}^{\infty}(W_N \cup V_N) \to \bigcup_{N=1}^{\infty}\Gamma_N$. $\hat{Q}_N$'s are as in Section 3 with resulting paths in Γ_N. P_N^u, $P_N'^u$, $\hat{P}_N^u$ and $\hat{P}_N'^u$ are regarded as probability measures on C.

In order to consider an almost sure limit, we shall couple walks on different pre-Sierpiński gaskets. Let

$$\Omega' = \{\omega = (\omega_0, \omega_1, \omega_2, \cdots) \ : \ \omega_0 = (O, a), \ \omega_N \in \Gamma_N, \ \omega_{N-1} \triangleright \omega_N, \ N \in \mathbb{N}\},$$

where $\omega_N \triangleright \omega_{N+1}$ means that there exists a $v \in W_{N+1} \cup V_{N+1}$ such that $Q_N v = \omega_N$ and $Lv = \hat{Q}_{N+1}v = \omega_{N+1}$. Namely, v is a path obtained by adding a finer, $2^{-(N+1)}$ - scale structure (not loopless yet) to ω_N, and erasing $2^{-(N+1)}$ - scale loops from v gives ω_{N+1}. We assumed $\omega_0 = (O, a)$ here, for we can deal with the case $\omega_0 = (O, b, a)$ in a similar way.

Define the projection onto the first $N+1$ elements by

$$\pi_N \omega = (\omega_0, \omega_1, \ldots, \omega_N).$$

For each $u \in [0,1]$, define a probability measure on $\pi_N \Omega'$ by

$$\tilde{P}_N[(\omega_0, \omega_1, \ldots, \omega_N)] = P_N^u[\ v : \ \hat{Q}_i v = \omega_i, i = 0, \ldots, N\ ],$$

where P_N^u is defined in Section 2. Although $\tilde{P}_N$ depends on u, we shall not write the u-dependence explicitly for simplicity. The following consistency condition is a direct consequence of the loop-erasing procedure:

$$\tilde{P}_N[(\omega_0, \omega_1, \ldots, \omega_N)] = \sum_{\omega'} \tilde{P}_{N+1}[(\omega_0, \omega_1, \ldots, \omega_N, \omega')], \tag{4.1}$$

where the sum is taken over all possible $\omega' \in \hat{W}_{N+1}$ such that $\omega_N \triangleright \omega'$.

By virtue of (4.1) and Kolmogorov's extension theorem for a projective limit, there is a probability measure P on $\Omega_0 = C^{\mathbb{N}} = C \times C \times \cdots$ such that

$$P[\ \Omega'\ ] = 1,$$

$$P \circ \pi_N^{-1} = \tilde{P}_N, \ N \in \mathbb{Z}_+,$$

where π_N denotes the projection onto the first $(N+1)$ elements here, too.

Let $Y_N : \ \Omega_0 \to \Gamma_N \subset C$ be the projection to the $(N+1)$-th component. We regard Y_N as an F-valued process $Y_N(\omega, t)$ on $(\Omega_0, \mathcal{B}, P)$, where $\mathcal{B}$ is the Borel algebra on Ω_0 generated by the cylinder sets. Then we have $P \circ Y_N^{-1} = \hat{P}_N^u$. For $N \geqq M$ and $\Delta \in \mathcal{T}_M$, denote the the path segment of Y_N in Δ by $Y_N|_\Delta$ as in (3.1).

For $w \in \bigcup_{N=1}^{\infty}\Gamma_N$ and $j = 1,2$, denote by $S_j^M(w)$ the number of 2^{-M}-triangles of Type j in $\sigma_M(w)$, and $\mathbf{S}^M(w) = (S_1^M(w), S_2^M(w))$. Note that if $w \in \Gamma_N$, then $\ell(w) = S_1^N(w) + 2S_2^N(w)$.

Let $\mathbf{S} = (S_1, S_2)$ and $\mathbf{S}' = (S_1', S_2')$ be $(\mathbb{Z}_+)^2$-valued random variables on $(\Omega_0, \mathcal{B}, P)$ with the same distributions as those of (S_1^1, S_2^1) under $\hat{P}_1^u$ and under $\hat{P}_1'^u$, respectively.

Proposition 4 *Fix arbitrarily $v \in \Gamma_M$, and let $\sigma_M(v) = (\Delta_1, \ldots, \Delta_k)$. For each i, $1 \leqq i \leqq k$, under the conditional probability $P[\ \cdot\ |Y_M = v]$, $\{\mathbf{S}^{M+N}(Y_{M+N}|_{\Delta_i}), \ N = 0, 1, 2, \cdots\}$ is a two-type supercritical branching process, with the types of children corresponding to the types of triangles. The offspring distributions born from a Type 1 triangle and from a Type 2 triangle are equal to those of $\mathbf{S}$ and $\mathbf{S}'$, respectively. If Δ_i is Type 1, the process initiates in state $(1,0)$, and if Δ_i is Type 2, in state $(0,1)$.*

(1) The generating functions for the offspring distributions are

$$E[\ x^{S_1}y^{S_2}\ ] = \hat{\Phi}(x,y),$$

$$E[\ x^{S'_1}y^{S'_2}\ ] = \hat{\Theta}(x,y),$$

where E is the expectation with regard to P.

(2) Let $\mathbf{M}$ be the mean matrix given by (3.7).
Then

$$E[\ \mathbf{S}^{M+N}(Y_{M+N}|_{\Delta_i})\ |\ Y_M = v\ ] = \mathbf{S}^M(v|_{\Delta_i})\mathbf{M}^N.$$

(3) $P[S_1 + S_2 \geqq 2] = P[S'_1 + S'_2 \geqq 2] = 1$ (non-singularity).

(4) $E[\ S_i \log S_i\ ] < \infty,\quad E[\ S'_i \log S'_i\ ] < \infty,\ i = 1,2.$

Proposition 4 suggests that we should consider the time-scaled processes:

$$X_N(\ \cdot\ ) = Y_N(\lambda^N\ \cdot\ ),\quad N \in \mathbb{Z}_+,$$

where λ is the larger eigenvalue of the mean matrix.

Proposition 5 *For $M \leqq N$, the following holds:*

$$\sigma_M(X_N) = \sigma_M(X_M) = \sigma_M(Y_M),\quad a.s.$$

and

$$X_N(T_i^{ex,M}(X_N)) = X_M(T_i^{ex,M}(X_M)) = Y_M(T_i^{ex,M}(Y_M)),\quad a.s. \tag{4.2}$$

Note that if $\sigma_M(X_N) = (\Delta_1, \cdots, \Delta_k)$, then

$$T_j^{ex,M}(X_N) = \lambda^{-N}\sum_{i=1}^{j}(S_1^N(X_N|\Delta_i) + 2S_2^N(X_N|\Delta_i)),\quad 1 \leqq j \leqq k.$$

Let $\mathbf{u} = {}^t(u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be the right and left positive eigenvectors associated with λ such that $|\mathbf{u}| = |\mathbf{v}| = 1$.

Proposition 6 *Fix arbitrarily $v \in \Gamma_M$, and let $\sigma_M(v) = (\Delta_1, \ldots, \Delta_k)$. For each i, $1 \leqq i \leqq k$, under the conditional probability $P[\ \cdot\ |Y_M = v]$, we have the following:*

(1) $\{\lambda^{-(M+N)}\mathbf{S}^{M+N}(X_{M+N}|_{\Delta_i}),\ N = 0,1,2,\ldots\}$ converges a.s. as $N \to \infty$ to a $\mathbb{R}^2$-valued random variable $\mathbf{S}^{*M,i} = (S_1^{*M,i}, S_2^{*M,i})$.

(2) $\{\mathbf{S}^{*M,i},\ i = 1, \cdots, k\}$ are independent.

(3) There are random variables B_1 and B_2 such that $\mathbf{S}^{*M,i}$ is equal in distribution to $\lambda^{-M}B_1\mathbf{v}$ if Δ_i is of Type 1, and equal in distribution to $\lambda^{-M}B_2\mathbf{v}$ if Δ_i is of Type 2.

(4)

$$P[B_i > 0] = 1,\quad E[B_i] = u_i,\quad i = 1,2.$$

B_1 and B_2 have strictly positive probability density functions.

(5) The Laplace transform of B_i, $i = 1, 2$

$$g_i(t) = E[\exp(-tB_i)], \quad t \in \mathbb{C}$$

are entire functions on $\mathbb{C}$ and are the solution to

$$g_1(\lambda t) = \hat{\Phi}(g_1(t), g_2(t)), \quad g_2(\lambda t) = \hat{\Theta}(g_1(t), g_2(t)), \; g_1(0) = g_2(0) = 1.$$

(1)–(4) in Proposition 6 are the straightforward consequences of general limit theorems for multi-type superbranching processes (Theorem 1 and Theorem 2 in V.6 of [1]). $P[B_i > 0] = 1$ is a consequence of $\hat{\Phi}$ and $\hat{\Theta}$ having no terms with degree smaller than 2. For the existence of the Laplace transform on the entire $\mathbb{C}$, we need careful study of the recursions. We omit the details here, since they are lengthy and similar to the proof of Proposition 4.5 in [9].

Let $T_i^{*M} = \sum_{j=1}^{i} (S_1^{*M,j} + 2S_2^{*M,j})$, then

$$\lim_{N\to\infty} T_j^{ex,M}(X_N) = T_j^{*M}. \tag{4.3}$$

By virtue of Proposition 5 and Proposition 6, we can prove the almost sure uniform convergence for X_N. The proof here closely follows the argument of [2].

Theorem 7 *X_N converges uniformly in t a.s. as $N \to \infty$ to a continuous process X.*

Proof. Choose $\omega \in \Omega'$ such that the following holds for all $M \in \mathbb{Z}_+$: $\lim_{N\to\infty} T_i^{ex,M}(X_N) = T_i^{*M}$ exists and $T_i^{*M} - T_{i-1}^{*M} > 0$ for all $1 \leqq i \leqq k$, where $k = k_M$ denotes the number of triangles in $\sigma_M(Y_M)$. Let $R = T_1^{*0} + \varepsilon$, where $\varepsilon > 0$ is arbitrary. It suffices to show that $X_N(\omega, t)$ converges uniformly in $t \in [0, R]$. In fact, if $t > R$, $X_N(t) = a$ for a large enough N.

Fix $M \in \mathbb{Z}_+$ arbitrarily. By expressing the arrival time at a as the sum of traversing times of 2^{-M}-triangles, we have $T_k^{ex,M}(X_N) = T_1^{ex,0}(X_N)$ a.s.. Letting $N \to \infty$, we have $T_k^{*M} = T_1^{*0}$ a.s..

The choice of ω implies that there exists an $N_1 = N_1(\omega) \in \mathbb{N}$ such that

$$\max_{1\leqq i\leqq k} |T_i^{ex,M}(X_N) - T_i^{*M}| \leqq \min_{1\leqq i\leqq k} (T_i^{*M} - T_{i-1}^{*M}), \quad |T_k^{ex,M}(X_N) - T_k^{*M}| < \varepsilon, \tag{4.4}$$

for $N \geqq N_1$.

If $0 \leqq t < T_k^{*M}$, then choose $j \in \{1, \cdots, k\}$ such that $T_{j-1}^{*M} \leqq t < T_j^{*M}$. Then (4.4) implies that $T_{j-2}^{ex,M}(X_N) \leqq t \leqq T_{j+1}^{ex,M}(X_N)$, for $N \geqq N_1$. Since Proposition 5 shows

$$X_N(T_j^{ex,M}(X_N)) = X_M(T_j^{ex,M}(X_M)), \tag{4.5}$$

for all N with $N \geqq M$, we have

$$|X_N(T_j^{ex,M}(X_N)) - X_N(t)| \leqq 3 \cdot 2^{-M}.$$

Otherwise, if $T_k^{*M} \leqq t \leqq T_k^{*M} + \varepsilon = R$, then let $j = k$. Since $T_{k-1}^{ex,M}(X_N) \leqq t$,

$$|X_N(T_j^{ex,M}(X_N)) - X_N(t)| \leqq 2^{-M}.$$

Therefore, if $N, N' \geqq N_1$, then for any $t \in [0, R]$,

$$\begin{aligned}
&|X_N(t) - X_{N'}(t)| \\
&\leqq \; |X_N(T_j^{ex,M}(X_N)) - X_N(t)| + |X_{N'}(T_j^{ex,M}(X_{N'})) - X_{N'}(t)| \\
&\quad\; +|X_N(T_j^{ex,M}(X_N)) - X_{N'}(T_j^{ex,M}(X_{N'}))| \\
&\leqq \; 6 \cdot 2^{-M},
\end{aligned}$$

where the third term in the middle part is 0 by (4.5). Since M is arbitrary, we have the uniform convergence.

$\square$

Proposition 6 (5) implies that $E[\exp tB_i] < \infty$ for $t > 0$, which leads to:

Proposition 8

$$P[\textit{There exist } t_0 < t_1 \textit{ such that } X(t) = X(t_0) \neq a \textit{ for all } t \in [t_0, t_1]] = 0.$$

The proof is similar to that in [6].

Proposition 9 *The following holds for all $M \in \mathbb{Z}_+$ almost surely:*

(1) $\sigma_M(X) = \sigma_M(X_M)$,

(2) $X(T_i^{*M}) = X_M(T_i^{ex,M}(X_M))$,

(3) Let $\sigma_M(X_M) = (\Delta_1, \cdots, \Delta_{k_M})$*. If* $T_{i-1}^{*M} < t < T_i^{*M}$*, then* $X(t) \in \Delta_i \backslash G_M$*, for all* $1 \leqq i \leqq k_M$*. In particular,* $T_i^{*M} = T_i^{ex,M}(X) = T_i^M(X)$*.*

Proof. (1) and (2) are direct consequences of Proposition 5, (4.3) and Theorem 7.

To prove (3), let $v_i = X(T_i^{*M})$, $i = 1, \cdots, k_M$ and we first prove that if $T_{i-1}^{*M} < t < T_i^{*M}$, then $X(t) \notin \{v_{i-1}, v_i\}$, by showing none of the following events A_j, $j = 1, 2, 3, 4$ has positive probability.

A_1 : There exists t_1, $T_{i-1}^{*M} < t_1 < T_i^{*M}$ such that $X(t) = v_i$ for all $t_1 < t \leqq T_i^{*M}$ holds for some $i \in \{1, \cdots, k_M\}$.

A_2 : There exists t_1, $T_{i-1}^{*M} < t_1 < T_i^{*M}$ such that $X(t) = v_{i-1}$ for all $T_{i-1}^{*M} \leqq t < t_1$ holds for some $i \in \{1, \cdots, k_M\}$.

A_3: There exist t_1 and t_2, $T_{i-1}^{*M} < t_1 < t_2 < T_i^{*M}$ such that $X(t_1) = v_i$ and $X(t_2) \neq v_i$ holds for some $i \in \{1, \cdots, k_M\}$.

A_4: There exist t_1 and t_2, $T_{i-1}^{*M} < t_1 < t_2 < T_i^{*M}$ such that $X(t_1) \neq v_{i-1}$ and $X(t_2) = v_{i-1}$ holds for some $i \in \{1, \cdots, k_M\}$.

Proposition 8 guarantees that $P[A_1] = P[A_2] = 0$. Since X is the uniform limit of a sequences of self-avoiding walks, we have $P[A_3] = P[A_4] = 0$.

Let $\sigma = (\Delta_1, \cdots, \Delta_{k_M})$ be a sequence such that $P[\sigma_M(X) = \sigma] > 0$. Let Δ_i be one of the triangles in σ, and denote the third vertex of Δ_i (neither the exit or entrance) by v_i^*. We prove that the probability that X hits v_i^* at some $T_{i-1}^{*M} < t < T_i^{*M}$ is zero. We can take a decreasing sequence of triangles $\{\Delta_i^{(K)}\}_{K=M}^{\infty}$ such that $\Delta_i^{(M)} = \Delta_i$, $\Delta_i^{(K)} \in \mathcal{T}_K$ (a 2^{-K}-triangle), $\Delta_i^{(K)} \supset \Delta_i^{(K+1)}$, $\bigcap_{K=M}^{\infty} \Delta_i^{(K)} = \{v_i^*\}$. Denote $\tilde{p} = \max\{\sum_{i=5}^{10} p_i, \sum_{i=5}^{10} q_i\} < 1$, where p_i and q_i are defined by (3.5). For any K, with $K \geqq M$, (1) implies

$$\begin{aligned} P[\ \Delta_i^{(K)} \in \sigma_K(X) \mid \sigma_M(X) = \sigma\ ] &= P[\ \Delta_i^{(K)} \in \sigma_K(X_K) \mid \sigma_M(X_K) = \sigma\ ] \\ &\leqq \tilde{p}^{K-M}. \end{aligned}$$

Thus it follows that

$$P[\ \Delta_i^{(K)} \in \sigma_K(X) \text{ for all } K \geqq M \mid \sigma_M(X) = \sigma\ ] = 0$$

and

$$P[\ \Delta_i^{(K)} \in \sigma_K(X) \text{ for all } K \geqq M \text{ for some } 1 \leqq i \leqq k_M \mid \sigma_M(X) = \sigma\ ] = 0,$$

therefore,

$$P[\ \Delta_i^{(K)} \in \sigma_K(X) \text{ for all } K \geqq M \text{ for some } i \in \{1, \cdots, k_M\}] = 0.$$

This implies that the probability that X hits any 'third' vertex of the triangles in its skeleton is zero. This completes the proof of (3).

□

This proposition further leads to ;

Theorem 10 *(1) X is almost surely self-avoiding in the sense that*

$$P[\ X(t_1) \neq X(t_2),\ 0 \leqq t_1 \leqq t_2 \leqq T_a(X)\ ] = 0,$$

where $T_a(X) = \inf\{t > 0 : X(t) = a\} = T_1^{*0}$.

(2) The Hausdorff dimension of the path $X([0, T_a(X)])$ is almost surely equal to $\log \lambda / \log 2$, which is a continuous function of u.

(1) is a consequence of Proposition 8 and Proposition 9. To calculate the Hausdorff dimension, we use the fact that if a path w is self-avoiding, then it holds that

$$\tilde{\sigma}_1(w) \supset \tilde{\sigma}_2(w) \supset \tilde{\sigma}_3(w) \supset \cdots \to w,$$

in the Hausdorff metric, where $\tilde{\sigma}_M(w)$ is the union of all the closed 2^{-M}-triangles in $\sigma_M(w)$. We can regard the path as a multi-type random fractal to obtain the Hausdorff dimension, applying Thoerem 4.3 in [4]. □

5 Path properties of the limit process

In this section we study some more sample path properties of the limit process. We assume $0 < u \leqq 1$, for the case of $u = 0$ is considered in [7]. We shall not explicitly write u-dependence as in the previous section.

Let

$$\nu = \nu(u) = \frac{\log 2}{\log \lambda}.$$

Recall, from Proposition 6 (5) that

$$g_i(t) = E[\ \exp(-tB_i)\ ], \quad i = 1, 2$$

satisfy the functional equations:

$$g_1(\lambda t) = \hat{\Phi}(g_1(t), g_2(t)),\ \ g_2(\lambda t) = \hat{\Theta}(g_1(t), g_2(t)).$$

Let

$$h_i(t) = -t^{-\nu} \log g_i(t).$$

The proof of the following proposition uses the explicit forms of $\hat{\Phi}$ and $\hat{\Theta}$, but it basically follows those of [2] and [14].

Proposition 11 *There exist positive constants C_1, C_2 and t_0 such that*

$$C_2 \leqq h_i(t) \leqq C_1 \quad i = 1, 2$$

hold for all $t \geqq t_0$.

Proof. We prove the upper bound for $i = 1$. Combining $g_1(\lambda t) = \hat{\Phi}(g_1(t), g_2(t))$ and the fact that $\hat{\Phi}(x, y)$ contains the term $p_1 x^2$, we have $g_1(\lambda t) \geqq p_1 g_1(t)^2$, which implies $h_1(\lambda t) \leqq \frac{a_2}{2}\, t^{-\nu} + h_1(t)$, where $a_2 = -\log p_1 > 0$. By induction, we have $h_1(\lambda^n t) \leqq a_2 t^{-\nu} + h_1(t)$, for any $t > 0$ and $n \in \mathbb{N}$. Fix $t_1 > 0$ arbitrarily. Since $h(t)$ is continuous for $t > 0$, $b_1 := \max_{t \in [t_1, \lambda t_1]} h_1(t)$ exists. For $t > \lambda t_1$, there is a positive integer m and $s \in (t_1, \lambda t_1]$ such that $t = \lambda^m s$. Then $h_1(t) = h_1(\lambda^m s) \leqq a_2 s^{-\nu} + h(s) \leqq a_2 t_1^{-\nu} + b_1 =: C_1$. Thus we have $h_1(t) \leqq C_1$ for any $t \geqq t_1$. The proof for $i = 2$ is similar, with the use of the term $q_4 y^2$ in $\hat{\Theta}(x, y)$. Note that $q_4 > 0$ for $u > 0$. Take the larger C_1.

To show the lower bound, first note that for $x, y \in [0,1]$, $\max\{\hat{\Phi}(x,y), \hat{\Theta}(x,y)\} \leqq \max\{x,^2, y^2\}$, which leads to $\hat{\Phi}(x,y) + \hat{\Theta}(x,y) \leqq 2(x+y)^2$. Let $g(t) := g_1(t) + g_2(t)$, then $g(\lambda t) \leqq 2g(t)^2$. $\tilde{h}(t) := -t^{-\nu} \log g(t)$ satisfies $\tilde{h}(\lambda t) \geqq (-(1/2)\log 2 - \log g(t))t^{-\nu} = -t^{-\nu}(1/2)\log 2 + \tilde{h}(t)$. By induction, we have $\tilde{h}(\lambda^n t) \geqq t^{-\nu}(-\log 2 - \log g(t))$. Since $-\log g(t) \to \infty$ as $t \to \infty$, we can take $t_2 > 0$ such that $-\log 2 - \log g(t) > 1$ for all $t \geqq t_2$, which implies $\tilde{h}(\lambda^n t) \geqq t^{-\nu}$ for all $t \geqq t_2$. In a similar way to the proof above, we can show that for any $t \geqq t_2$, $\tilde{h}(t) \geqq (1/2)t_2^{-\nu} =: C_2$, thus $h_i(t) = -t^{-\nu}\log g_i(t) \geqq -t^{-\nu}\log g(t) = \tilde{h}(t) \geqq C_2$ holds for both $i = 1, 2$. Let $t_0 = \max\{t_1, t_2\}$. □

We now use a Taubelian theorem of exponential type. The following theorem, Corollary A.17 from [5] has a most suitable form for our purpose.

Theorem 12 *Assume P is a Borel probability measure supported on $[0,\infty)$, and denote its Laplace transform by*

$$g(s) = \int_0^\infty e^{-s\xi} P[d\xi], \quad s > 0.$$

If there are constants $C_1 > 0$, $C_2 > 0$ and $0 < \nu < 1$ such that

$$-C_1 \leqq \varliminf_{s\to\infty} s^{-\nu} \log g(s) \leqq \varlimsup_{s\to\infty} s^{-\nu} \log g(s) \leqq -C_2,$$

then there exist $C_3 > 0$ and $C_4 > 0$ such that

$$-C_3 \leqq \varliminf_{x\to 0} x^{\nu/(1-\nu)} \log P[[0,x]] \leqq \varlimsup_{x\to 0} x^{\nu/(1-\nu)} \log P[[0,x]] \leqq -C_4, \quad x > 0.$$

Let $W_i = (v_1 + 2v_2)B_i,\ i = 1,2$, where $\mathbf{v} = (v_1, v_2)$ is the positive left eigenvector corresponding to λ introduced just before Proposition 6 in Section 4. Then Proposition 11 and Theorem 12 lead to

Corollary 13 *There exist positive constants C_5, C_6, and x_0 such that*

$$e^{-C_5 x^{-\frac{\nu}{1-\nu}}} \leqq P[\ W_i \leqq x\ ] \leqq e^{-C_6 x^{-\frac{\nu}{1-\nu}}}, \quad i = 1,2$$

hold for any $x \leqq x_0$.

Proposition 14 *There exist positive constants C_7, C_8 and K such that*

$$e^{-C_7(\delta t^{-\nu})^{1/(1-\nu)}} \leqq P[\ |X(t)| \geqq \delta\ ] \leqq P[\ \sup_{0\leqq s\leqq t} |X(s)| \geqq \delta\ ] \leqq e^{-C_8(\delta t^{-\nu})^{1/(1-\nu)}}, \quad i = 1,2$$

hold for $\delta t^{-\nu} \geqq K$.

Proof. For an arbitrarily given $0 < \delta < 1$, take $N \in \mathbb{N}$ such that $2^{-N} < \delta \leqq 2^{-N+1}$ holds. Recall that if Δ_1, the first element of $\sigma_N(X)$, is of Type 1, $T_1^{ex,N}(X)$ has the same distribution as that of $\lambda^{-N}W_1$, and if of Type 2, the same distribution as that of $\lambda^{-N}W_2$. For $i = 1,2$ denote by A_i the event that Δ_1, is of Type i.

For the upper bound, since $\sup_{0\leqq s\leqq t} |X(s)| \geqq \delta$ implies $T_1^{ex,N}(X) < t$,

$$\begin{aligned}
P[\ \sup_{0\leqq s\leqq t} |X(s)| \geqq \delta\ ] &\leqq P[\ T_1^{ex,N}(X) < t\ ] \\
&= P[\ W_1 < \lambda^N t\ ]\ P[\ A_1\ ] + P[\ W_2 < \lambda^N t\ ]\ P[\ A_2\ ] \\
&\leqq e^{-C_6(\lambda^N t)^{-\frac{\nu}{1-\nu}}} \\
&\leqq e^{-C_8(\delta t^{-\nu})^{1/(1-\nu)}},
\end{aligned}$$

where we assumed that $\lambda^N t \leqq x_0$ in the second inequality and set $C_8 = 2^{-1/(1-\nu)} C_6$.

For the lower bound, since $T_1^{ex,N-1} < t$ implies $|X(t)| \geqq \delta$, we can show that there exists a $C_7 > 0$ such that

$$P[\ |X(t)| \geqq \delta\ ] \geqq e^{-C_7(\delta t^{-\nu})^{1/(1-\nu)}}$$

holds for $\lambda^{N-1} t \leqq x_0$. Take $K = 2x_0^{-\nu}$. □

Theorem 15 *For any $p > 0$, there are positive constants C_9 and C_{10} such that*

$$C_9 \leqq \varliminf_{t \to 0} \frac{E[|X(t)|^p]}{t^{p\nu}} \leqq \varlimsup_{t \to 0} \frac{E[|X(t)|^p]}{t^{p\nu}} \leqq C_{10}.$$

Proof. Proposition 14 implies that the following holds for large enough t:

$$\begin{aligned}
\frac{1}{p} E[|X(t)|^p] &= \int_0^1 \delta^{p-1} P_i[|X(t)| \geqq \delta]\ d\delta \geqq \int_{Kt^\nu}^1 \delta^{p-1} P_i[|X(t)| \geqq \delta]\ d\delta \\
&\geqq \int_{Kt^\nu}^1 \delta^{p-1} e^{-C_7(\delta t^{-\nu})^{1/(1-\nu)}} d\delta = t^{p\nu} \int_K^{t^{-\nu}} y^{p-1} e^{-C_7 y^{1/(1-\nu)}}\ dy \\
&\geqq \frac{1}{2} t^{p\nu} \int_K^\infty y^{p-1} e^{-C_7 y^{1/(1-\nu)}}\ dy = C_9 t^{p\nu},
\end{aligned}$$

$$\begin{aligned}
\frac{1}{p} E[|X(t)|^p] &= \int_0^{Kt^\nu} \delta^{p-1} P_i[|X(t)| \geqq \delta]\ d\delta + \int_{Kt^\nu}^1 \delta^{p-1} P_i[|X(t)| \geqq \delta]\ d\delta \\
&\leqq \int_0^{Kt^\nu} \delta^{p-1} d\delta + t^{p\nu} \int_K^\infty y^{p-1} e^{-C_8 y^{1/(1-\nu)}}\ dy \int_{Kt^\nu}^1 \delta^{p-1} = C_{10} t^{p\nu}.
\end{aligned}$$

□

Corollary 13 and Proposition 14 lead to a law of the iterated logarithm. Since the argument is similar to that in [3], we just give the statement below:

Theorem 16 *There are positive constants C_{11} and C_{12} such that*

$$C_{11} \leqq \varlimsup_{t \to 0} \frac{|X(t)|}{\psi(t)} \leqq C_{12}, \ \ a.s.,$$

where $\psi(t) = t^\nu (\log\log(1/t))^{1-\nu}$.

6 Conclusion and remarks

We constructed a one-parameter family of self-avoiding walks that interpolates the SAW and the LERW on the Sierpiński gasket, and proved that the scaling limit exists. The exponent that governs the short-time behavior and equals to the reciprocal of the path Hausdorff dimension is a continuous function of the parameter. Our construction has proved that the ELLF method does work for non-Markov random walks as well as the simple random walk.

Although we restricted ourselves to $u \in [0, 1]$ above, all the results hold also for $u > 1$, that is, for self-attracting walks. By numerical calculations we observe that λ is a decreasing function of u and conjecture that as $u \to \infty$, $x^* = \lim_{u\to\infty} u x_u$, $p_i^* = \lim_{u\to\infty} p_i(x_u, u)$ and $q_i^* = \lim_{u\to\infty} q_i(x_u, u)$ exist with $x^* \sim 0.351$, $p_1^* \sim 0.206$, $p_2^* \sim 0.124$, $p_3^* \sim 0.206$, $p_4^* \sim 0.352$,

$p_5^* \sim 0.083$, $p_6^* \sim 0$, $p_7^* \sim 0.029$, $q_1^* \sim 0.345$, $q_2^* \sim 0.034$, $q_3^* \sim 0.242$, $q_4^* \sim 0.097$, $q_5^* \sim 0.208$, $q_7^* \sim 0.073$ and $q_i^* \sim 0$ otherwise.

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Appendix

Define for $i = 1, 2, \cdots, 10$

$$p_i(x,u) = \sum\nolimits_i u^{N(w)+M(w)} x^{\ell(w)-1}, \quad q_i(x,u) = \sum\nolimits_i' u^{N(w)+M(w)} x^{\ell(w)-1}, \; x, u \geqq 0$$

where the sum $\sum_i$ is taken over all $w \in W_1$ such that $Lw = w_i^*$ and $\sum_i'$ over all $w \in V_1$ such that $Lw = w_i^*$. Substituting $x = x_u$, we have

$$p_i = p_i(x_u, u) = \hat{P}_1^u[w_i^*], \; q_i = q_i(x_u, u) = \hat{P}_1'^u[w_i^*].$$

Let U_1 be a set of single loops formed at O on F_1:

$$U_1 = \{ \; w = (w(0), w(1), \cdots, w(n)) : \; w(0) = w(n) = O, \; w(i) \in G_1 \setminus G_0, \; 1 \leqq i \leqq n-1,$$
$$\{w(i), w(i+1)\} \in E_1, \; 0 \leqq i \leqq n-1, \; n \in \mathbb{N} \; \}.$$

and define $N(w)$ by (2.1). Define

$$\Xi = \Xi(x,u) = \sum_{w \in U_1} u^{N(w)} \, x^{\ell(w)}, \; x, u \geqq 0.$$

We obtain the explicit form (Θ in [3]) as follows:

$$\Xi(x,u) = \frac{2ux^2}{(1+ux)(1-2ux)} \{1 + 2(1-u^2)x^2 - 2(1-u)^2 ux^3\}.$$

We show the explicit forms of $p_i(x,u)$ and $q_i(x,u)$ below. Each factor in these expressions represents a particular part of paths. The common factor $1/(1-2u\Xi)$ comes from the sum over all the possible loops formed at O. In the lengthy expression of $q_2'(x,u)$, the first term is related to those paths with loops that are formed at $(1/2, 0)$ and include b. The factor $x^2 \left(1 + ux + \dfrac{u^2x^2}{(1+ux)(1-2ux)} \{2(u^2-u+1)x+3\}\right)$ represents the part from the last hit at O followed immediately by a step to $(1/2,0)$ then to the first hit of b. The factor $1/(1-2u\Xi)$ stands for the sum over all the possible loops formed at b. $x^2 \left(1 + \dfrac{u^2x(1+2x)}{(1+ux)(1-2x)}\right)$ corresponds to the trip back from b to $(1/2,0)$ followed immediately by a step to $(1/4, \sqrt{3}/4)$, and $\left(1 + \dfrac{u^3x^2}{1-u^2x^2} + \dfrac{u^3x^4}{(1-u^2x^2)^2} \dfrac{1}{1-u(\ell+\Xi)}\right)$ concerns the loops formed at $(1/4, \sqrt{3}/4)$. The second term is related to paths whose first hit to b occurs in a loop formed at $(1/4, \sqrt{3}/4)$.

$$p_1(x,u) = \frac{x}{1-2u\Xi} \left(1 + \frac{u^2x^2\{(1-u)^2x+2\}}{(1+ux)(1-2ux)}\right).$$

$$p_2(x,u) = \frac{ux^2}{1-2u\Xi} \left(1 + \frac{u(1+u)x^2}{(1+ux)(1-2ux)}\right) \left(1 + \frac{u^3x^2}{1-u^2x^2}\right)$$

$$p_3(x,u) = \frac{ux^2}{1-2u\Xi} \left(1+\frac{u^2(1+u)x^2}{(1+ux)(1-2ux)}\right) \left(1+\frac{ux^2}{1-u^2x^2}\right).$$

$$p_4(x,u) = \frac{u^3x^3}{(1-2u\Xi)(1-u^2x^2)} \left(1+\frac{u(1+u)x^2}{(1+ux)(1-2ux)}\right).$$

$$p_5(x,u) = \frac{u^2x^3}{(1-2u\Xi)(1-u^2x^2)} \left(1+\frac{u^2(1+u)x^2}{(1+ux)(1-2ux)}\right).$$

$$p_6(x,u) = \frac{u^2x^3}{(1-2u\Xi)(1-u^2x^2)} \left(1+\frac{u(1+u)x^2}{(1+ux)(1-2ux)}\right).$$

$$p_7(x,u) = \frac{x^2}{1-2u\Xi} \left(1+\frac{u^2(1+u)x^2}{(1+ux)(1-2ux)}\right) \left(1+\frac{u^3x^2}{1-u^2x^2}\right)$$

$$p_8(x,u) = p_9(x,u) = p_{10}(x,u) = 0.$$

Let

$$\ell(x,u) = ux^2 + \frac{ux^4}{1-u^2x^2}, \quad \Sigma(x,u) = \frac{2ux^2}{1-ux}$$

and

$$q_i(x,u) = \frac{1}{x^2}\frac{1}{1-2u\Xi}\; q_i'(x,u), \quad i = 1, \cdots, 10.$$

Then

$$q_1'(x,u) = \frac{(1+u)^2x^6}{1-2u\Xi}\left(1+\frac{u^2x(1+2x)}{(1+ux)(1-2ux)}\right)^2,$$

$$\begin{aligned} q_2' = x^5 &\left(1+ux+\frac{u^2x^2}{(1+ux)(1-2ux)}\{2(u^2-u+1)x+3\}\right) \frac{1}{1-2u\Xi}\left(1+\frac{u^2x(1+2x)}{(1+ux)(1-2x)}\right) \\ &\times\left(1+\frac{u^3x^2}{1-u^2x^2}+\frac{u^3x^4}{(1-u^2x^2)^2}\frac{1}{1-u(\ell+\Xi)}\right) \\ &+\frac{u^3x^7}{(1-u^2x^2)^2(1-u(\ell+\Xi))}\left(1+\frac{u(1+u)x^2}{(1+ux)(1-2ux)}\right), \end{aligned}$$

$$\begin{aligned} q_3'(x,u) = &\frac{2u(1+u)x^7}{1-2u\Xi}\left(1+\frac{u^2x(1+2x)}{(1+ux)(1-2ux)}\right)^2 \\ &\times\left\{u+\frac{u^2x^2}{1-u^2x^2}+\frac{u}{1-u(\Sigma+\Xi)}\frac{x^2}{1-ux}\left(1+\frac{u^2x}{1-ux}\right)\right\} \\ &+\frac{x^5}{1-u(\Sigma+\Xi)}\left(1+\frac{u^2(1+u)x^2}{(1+ux)(1-2ux)}\right)\frac{1}{1-ux}\left(1+\frac{u^2x}{1-ux}\right), \end{aligned}$$

$$q_4'(x,u) = \frac{u^2x^6}{1-2u\Xi}\left(1+ux+\frac{u^2x^2}{(1+ux)(1-2ux)}\left\{2(u^2-u+1)x+3\right\}\right)\left(1+\frac{u^2x(1+2x)}{(1+ux)(1-2ux)}\right)$$

$$\times\frac{1}{1-u^2x^2}\left(1+\frac{u^2x^4}{(1-u^2x^2)(1-u(\ell+\Xi))}\right)\left(1+\frac{x^2}{1-u(ux^2+\Xi)}\right)$$
$$+\frac{u^4x^8}{(1-u^2x^2)^2(1-u(\ell+\Xi))}\left(1+\frac{u(1+u)x^2}{(1+ux)(1-2ux)}\right)\left(1+\frac{x^2}{1-u(ux^2+\Xi)}\right)$$
$$+\frac{u^2x^6}{(1-u^2x^2)(1-u(ux^2+\Xi))}\left(1+\frac{u(1+u)x^2}{(1+ux)(1-2ux)}\right),$$

$$q_5'(x,u)=\frac{u^3x^6}{1-2u\Xi}\left(1+ux+\frac{u^2x^2}{(1+ux)(1-2ux)}\left\{2(u^2-u+1)x+3\right\}\right)$$
$$\times\left(1+\frac{ux(1+2x)}{(1+ux)(1-2ux)}\right)\left\{\left(1+\frac{ux^2}{1-u^2x^2}\right)\frac{ux^2}{(1-u^2x^2)(1-u(\ell+\Xi))}+\frac{1}{1-u^2x^2}\right\}$$
$$+u^2x^6\left(1+\frac{u^2(1+u)x^2}{(1+ux)(1-2ux)}\right)\left(1+\frac{ux^2}{1-u^2x^2}\right)\frac{1}{1-u(\ell+\Xi)}\ \frac{1}{1-u^2x^2}$$

$$q_6'(x,u)=2u(1+u)x^8\left(1+\frac{u^2x(1+2x)}{(1+ux)(1-2ux)}\right)^2\frac{1}{1-2u\Xi}$$
$$\times\left\{\left(1+\frac{u^2x}{1-ux}\right)\frac{u}{1-u(\Sigma+\Xi)}\frac{ux^2}{1-ux}+\frac{u}{1-u^2x^2}\right\}\left(1+\frac{u^2x^2}{1-u(ux^2+\Xi)}\right)$$
$$+u^2x^6\left(1+\frac{u(1+u)x^2}{(1+ux)(1-2ux)}\right)\left(1+\frac{u^2x}{1-ux}\right)\frac{1}{1-u(\Sigma+\Xi)}\frac{1}{1-ux}\left(1+\frac{u^2x^2}{1-u(ux^2+\Xi)}\right)$$
$$+u^3x^6\left(1+\frac{u(1+u)x^2}{(1+ux)(1-2ux)}\right)\frac{1}{1-u^2x^2}\frac{1}{1-u(ux^2+\Xi)},$$

$$q_7'(x,u)=\frac{ux^5}{1-2u\Xi}\left(1+ux+\frac{u^2x^2}{(1+ux)(1-2ux)}\left\{2(u^2-u+1)x+3\right\}\right)\left(1+\frac{ux(1+2x)}{(1+ux)(1-2ux)}\right)$$
$$\times\left\{\left(1+\frac{ux^2}{1-u^2x^2}\right)\frac{u^2x^2}{1-u^2x^2}\ \frac{1}{1-u(\ell+\Xi)}+\left(1+\frac{u^3x^2}{1-u^2x^2}\right)\right\}$$
$$+ux^5\left(1+\frac{u^2(1+u)x^2}{(1+ux)(1-2ux)}\right)\left(1+\frac{ux^2}{1-u^2x^2}\right)\frac{1}{1-u(\ell+\Xi)}\ \frac{1}{1-u^2x^2},$$

$$q_8'(x,u)=\left\{u^2x^4\left(1+ux+\frac{u^2x^2}{(1+ux)(1-2ux)}\left\{2(u^2-u+1)x+3)\right\}\right)\frac{1}{1-2u\Xi}\right.$$
$$\times\left(1+ux+x^2\frac{(4u^2-2u)x+u^2+2}{(1+ux)(1-2ux)}\right)$$
$$\left.+x^2\left(1+\frac{u^2x^2}{(1+ux)(1-2ux)}\left\{(1-u)^2x+2\right\}\right)\right\}\frac{1}{1-u(\ell+\Xi)}\ \frac{x^2}{1-u^2x^2},$$

$$q_9'(x,u)=uxq_8'(x,u),$$

$$q_{10}'(x,u)=2u(1+u)x^6\left(1+\frac{u^2x(1+2x)}{(1+ux)(1-2ux)}\right)^2\frac{1}{1-2u\Xi}$$

$$\times \left\{ x \left(1 + \frac{u^2 x}{1-ux}\right) \frac{1}{1-u(\Sigma+\Xi)} \frac{u^2x^2}{1-ux} + x + \frac{u^3x^3}{1-u^2x^2} \right\} \; u \; \frac{x^2}{1-u(ux^2+\Xi)}$$
$$+ux^2 \left(1 + \frac{u(1+u)x^2}{(1+ux)(1-2ux)}\right) \left\{ x \left(1 + \frac{u^2 x}{1-ux}\right) \frac{1}{1-u(\Sigma+\Xi)} \frac{u^2x^2}{1-ux} + x + \frac{u^3x^3}{1-u^2x^2} \right\}$$
$$\times \frac{x^2}{1-u(ux^2+\Xi)}.$$

Using MATHEMATICA, we have confirmed that as functions of x and u, the following holds:

$$\sum_{i=1}^{10} p_i(x,u) = \Phi(x,u)/x, \quad \sum_{i=1}^{10} q_i(x,u) = \Phi(x,u)^2/x^2,$$

as required by the definitions of $\hat{P}_1^u$ and $\hat{P}_1'^u$, where $\Phi(x,u)$ is defined in (2.3).

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