Preface

Energy of knots was introduced to produce an "optimal embedding" of a knot, a beautiful knot which would represent its knot type. The basic philosophy due to Fukuhara and Sakuma independently is as follows.

Suppose there is a non-conductive knotted string which is charged uniformly in a non-conductive viscous fluid. Then it might evolve to decrease its electrostatic energy without intersecting itself because of Coulomb's repulsive force until it comes to a critical point of the energy. Hence we might be able to define an "optimal embedding" of a knot by an "energy minimizer", which is an embedding that attains the minimum energy within its isotopy class. Thus our motivational problem can be stated as:

Can we define an "energy" on the space of knots so that its gradient flow can evolve a knot to an "optimal embedding" while preserving its isotopy type?

For this purpose, our energy should blow up if a knot degenerates to a "singular knot" with double points since a knot might change its knot type through a crossing change.

The first example of such an energy, $E^{(2)}$, for smooth knots was defined as the renormalization of a modified electrostatic energy of charged knots under the assumption that the absolute value of the repulsive force between a pair of unit point charges is proportional to the inverse cube of the distance.

In Part 1 we introduce several types of the knot energy and study the problem of whether or not there is an "energy minimizer" in each knot type. We first give a family of repulsive energies, which are obtained as the renormalization of modified electrostatic energy of charged knots. Then the answer to the above problem may depend on the following three conditions; the power exponent with which the distance between a pair of points on a knot are integrated, the primeness of a knot, which is a topological condition, and the metric of the ambient space, which is a geometric condition. We then introduce geometrically defined knot energies of another kind, such as ropelength. Roughly speaking they measure how much a knot can be fattened, or how tight a rope can be knotted.

The interest in knot energies arose naturally when we regarded the knots as real objects. They have concrete backgrounds and applications in various fields of natural science. We introduce some results of numerical experiments.

In Part 2 we study conformal geometry. The results in Part 2 were obtained in joint work with Rémi Langevin, summarized in [LO1], from which we cite frequently. The first example of the knot energy, $E^{(2)}$, was eventually proved to be invariant under Möbius transformations of $\mathbb{R}^3 \cup \{\infty\}$ by Freedman, He, and Wang. Allowing the action of a large group as the Möbius group has both advantages and disadvantages. If a knot is an energy minimizer then its image by any Möbius transformation is again an energy minimizer of the same knot type (or possibly, its mirror image). On the other hand, conformal geometry could provide a new viewpoint to knot theory. We use the set of spheres in S^3, which forms a 4-dimensional hyperbolic hypersurface in Minkowski space $\mathbb{R}^{4,1}$. Using a sphere that is tangent to a knot K at two points we introduce a complex valued (meromorphic) 2-form on $K \times K$ which will be called the infinitesimal cross ratio. This 2-form will give another interpretation of $E^{(2)}$ and also it can express other conformally invariant energies. Finally an energy defined from an integral geometric viewpoint will be introduced.

The energies presented in this book are defined geometrically and measure the complexity of embeddings. The study of our these subjects is called physical knot theory.

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Some of the sections in Part 2, in particular Section 13.6, are based on the manuscript for [LO1] by Langevin.