Research Abstract

Jun O'Hara starded the study of *energy of knots*. It gave rise to *geometric knot theory*, in which we study functionals to measure how complicated a knot is and look for "optimal knots" with respect to those functionals. The field involves not only branches of mathematics but also bio-chemstry and soft matter physics. It has been studied actively for recent two decades. Jun O'Hara e began joint work with Rémi Langevin (Université de Bourgogne) in 1999, studying various functionals on the space of knots (or that of links or surfaces) from a viewpoint of conformal or Möbius geometry.

1. Energy of knots.

The motivational problem, which was proposed by Fukuhara and Sakuma independently, is:

Problem Give a functional e on the space of knots $\mathcal{K}$ which satisfies the following conditions:

1. Let $[K]$ denote an isotopy class which contains a knot K. Define the energy of an isotopy class by $e([K]) = \inf\limits_{K' \in [K]} e(K')$.
2. If a knot K_0 attains the minimum value of the functional e within its isotopy class, i.e. if $e(K_0) = e([K_0])$, we call K_0 an *e-minimizer* of the isotopy class $[K_0]$. An minimizer should be good-looking.
3. Better to produce an e-minimizer in each isotopy class.

For this purpose, Jun O'Hara defined the energy E (*Energy of a knot*, Topology 30 (1991)). It can be obtained by renormalizing generalized electrostatic energy of charged knots. The value of it blows up as a knot degenerates to a singular knot with double points. When a functional on the space of knots has this property, we will call it an *energy functional of knots.* Later on, the energy E was proved to be invariant under Möbius transformations by Freedman, He, and Wang (and hence is sometimes called Möbius energy). Using this property they gave a partial affirmative answer to the problem above, namely, they showed that every isotopy class of a *prime* knot has an E-minimizer. (On the other hand, through numerical experiments, Kusner and Sullivan conjectured that there would be no E-minimizers in any isotopy class of a composite knot.)

Jun O'Hara defined other types of knot energies that produce energy minimizers for each isotopy class. These energies are not invariant under Möbius transformations. He also considered the case when the ambient space is a sphere or a hyperbolic space.

Since Jun O'Hara defined the first example of knot energy, various kinds of generalization have been studied by Kusner, Lin, Brylinski, Buck, Simon, et al. and numerical experiments have been executed by many groups.

Jun O'Hara also defined geometric quantities for knots which can be defined using distances between pairs of points on the knots.

2. Conformal geometry.

Rémi Langevin and Jun O'Hara defined the *infinitesimal cross ratio* of a knot in *Conformally invariant energies of knots*, J. Institut Math. Jussieu 4 (2005) as follows. Let x and y be a pair of points on a knot K. Let Σ be a sphere which is tangent to the knot at x and y. By identifying

Σ with a complex sphere, we can consider four points x, $x+dx$, y, and $y+dy$ as four complex numbers. Let Ω be the cross ratio. It can be considered as a complex valued 2-form on the two point configuration space $K \times K \setminus \triangle$. We call it the infinitesimal cross ratio.

The real part of it is equal to the pull-back by the inclusion map $K \times K \setminus \triangle \hookrightarrow S^3 \times S^3 \setminus \triangle$ of the canonical symplectic form of the cotangent bundle T^*S^3 under the identification of T^*S^3 with $S^3 \times S^3 \setminus \triangle$. Therefore, it is an exact form. On the other hand, the imaginary part of the infinitesimal cross ratio cannot be expressed in this way.

The energy of knots E can be expressed as the integral of the difference of the absolute value and the real part of the infinitesimal cross ratio on $K \times K \setminus \triangle$: $E(K) = \displaystyle\iint_{K\times K\setminus\triangle} |\Omega| - \Re\mathfrak{e}\,\Omega$.

The integration of the absolute value of the imaginary part of the infinitesimal cross ratio is also an energy functional of knots.

These results can be found in the book by Jun O'Hara, *Energy of knots and conformal geometry*, Series on Knots and Everything Vol. 33, World Scientific, Singapore, 304 pages. (2003).

Rémi Langevin and Jun O'Hara study the set of hyperspheres in the study of curves and surfaces from a conformal geometric viewpoint. Let $\mathcal{S}(q,n)$ $(0 \le q < n)$ be the set of oriented q-dimensional sphere in S^n or $\mathbb{R}^n$. It can be identified with the Grassmann manifold $SO(n+1,1)/SO(n-q) \times SO(q+1,1)$ which consists of oriented $q+2$-dimensional subspaces in the Minkowski space $\mathbb{R}^{n+2}_1 = \mathbb{R}^{n+1,1}$ that intersect the light cone transversely. It is a $(q+2)(n-q)$ dimensional pseudo-Riemannian (or semi-Riemannian) manifold with index $n-q$. Especially, when $q=n-1$, $\mathcal{S}(n-1,n)$ can be identified with de Sitter space, which is hyperbolic hypersurface with one sheet in $\mathbb{R}^{n+2}_1$. When $q=0$, $\mathcal{S}(0,n)$ has a natural symplectic structure since it can be identified with the cotangent bundle T^*S^n. These structures, the pseudo-Riemannian structure and the symplectic structure when $q=0$ are invariant under Möbius transformations.

Here are some of the applications to the theory of curves and surfaces:

(i) Let K be a knot, and $\mathcal{S}(K)$ be the set of oriented spheres that intersect K in more than or equal to four points, where the number of the intersection points is counted with multiplicity. Then the measure (with multiplicity) of $\mathcal{S}(K)$ with respcet to the pseudo-Riemannian structure of $\mathcal{S}(n-1,n)$ turns out to be an energy functional of knots. Jun O'Hara obtained a formula which expresses the measure above mentioned in terms of the infinitesimal cross ratio.

(ii) Fialkow showed in 1942 that a curve C in $\mathbb{R}^3$ is determined by the conformal arc-length, conformal curvature, and conformal torsion up to the action of Möbius group. The set of osculating circles to C forms a *null* curve γ in the set of oriented circles in S^3. Rémi Langevin and Jun O'Hara defined the $\frac{1}{2}$*-dimensional arc-length element of a null curve* and showed that that of γ is equal to the conformal arc-length of C.

3. Configuration spaces.

Jun O'Hara is also interested in the configuration space of polygons or linkages.