Structure of near-threshold hadrons

Tetsuo Hyodo
Yukawa Institute for Theoretical Physics, Kyoto Univ.

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Introduction: structure of hadrons

Compositeness of hadrons and near-threshold bound states

S. Weinberg, Phys. Rev. 137, B672 (1965);

Near-threshold resonances


Near-threshold mass scaling


(Three-body case)
Introduction: structure of hadrons

Exotic structure of hadrons

Various excitations of baryons

**conventional**

- An internal excitation

**exotic**

- Multiquark hadronic molecule

**Physical state:** superposition of 3q, 5q, MB, ...

\[
|\Lambda(1405)\rangle = N_{3q} |uds\rangle + N_{5q} |uds \ q\bar{q}\rangle + N_{\bar{K}N} |\bar{K}N\rangle + \cdots
\]

Is this [relevant strategy](#)?
Introduction: structure of hadrons

Ambiguity of definition of hadron structure

Decomposition of hadron “wave function”

\[ | \Lambda(1405) \rangle = N_{3q} | uds \rangle + N_{5q} | uds \, q\bar{q} \rangle + N_{\bar{K}N} | \bar{K}N \rangle + \cdots \]

- \( N_X \neq \) probability?

- 5q v.s. MB: double counting (orthogonality)?

\[ \langle udsq\bar{q} | \bar{K}N \rangle \neq 0 \]

- 3q v.s. 5q: not clearly separated in QCD

\[ \langle uds | udsq\bar{q} \rangle \neq 0 \]

- hadron resonances: unstable, finite decay width

\[ | \Lambda(1405) \rangle = ? \]

What is the suitable basis to classify the hadron structure?
**Strategy**

**Elementary/composite** nature of bound states near the lowest energy two-body threshold

- **elementary**
  - $6q$ for deuteron
  - $c\bar{c}$ for $X(3872)$

- **composite**
  - $NN$ for deuteron
  - $\bar{D}D^*$ for $X(3872)$

- orthogonality $\leftarrow$ eigenstates of bare Hamiltonian
- normalization $\leftarrow$ eigenstate of full Hamiltonian
- model dependence $\leftarrow$ low energy universality

* Basis must be asymptotic states (in QCD, hadrons).
* “Elementary” stands for any states other than two-body composite (missing channels, CDD pole, …).
Compositeness of hadrons and near-threshold bound state

**Formulation**

Coupled-channel Hamiltonian (bare state + continuum)

\[
\begin{pmatrix}
M_0 & \hat{V} \\
\hat{V} & \frac{p^2}{2\mu} (1 + \hat{V}_{sc})
\end{pmatrix}
\begin{pmatrix}
\Psi \\
\Psi
\end{pmatrix} = E
\begin{pmatrix}
\Psi \\
\Psi
\end{pmatrix},
\]

\[
\begin{pmatrix}
\Psi \\
\Psi
\end{pmatrix} = \left(\frac{c(E)}{\chi_E(p)}\right)
\begin{pmatrix}
\psi_0 \\
p
\end{pmatrix}
\]


Elementariness by field renormalization constant

- Bound state normalization + completeness relation

\[
\langle \Psi | \Psi \rangle = 1
\]

\[
1 = |\psi_0\rangle \langle \psi_0 | + \int d^3q|q\rangle \langle q |
\]

\[
1 = \left| \langle \Psi | \begin{pmatrix} \psi_0 \\ 0 \end{pmatrix} \rangle \right|^2 + \int d^3q \left| \langle \Psi | \begin{pmatrix} 0 \\ q \end{pmatrix} \rangle \right|^2 = Z + X
\]

\[
Z, X : \text{real and nonnegative} \rightarrow \text{probabilistic interpretation}
\]
Compositeness of hadrons and near-threshold bound state

**Weak binding limit**

In general, $Z$ is determined by the potential $\hat{V}$.

$$Z(B) = \frac{1}{1 - \frac{d}{dE} \int \frac{1}{E-q^2/(2\mu)+i0^+} d^3q \bigg|_{E=-B}} \equiv \frac{1}{1 - \Sigma'(-B)}$$

- $Z$ is a model-(scheme-)dependent (c.f. potential)

$Z$ of weakly-bound ($R \gg R_{\text{typ}}$) s-wave state $\leftarrow$ observables.

S. Weinberg, Phys. Rev. 137, B672 (1965);

$$a = \frac{2(1 - Z)}{2 - Z} R + O(R_{\text{typ}}), \quad r_e = \frac{-Z}{1 - Z} R + O(R_{\text{typ}}),$$

$\mathbf{a}$ : scattering length, $r_e$ : effective range
$R = (2\mu B)^{-1/2}$ : radius $\leftarrow$ binding energy
$R_{\text{typ}}$ : typical length scale of the interaction

- can be explicitly derived by the expansion of the amplitude

Compositeness of hadrons and near-threshold bound state

Model independence in weak binding limit

Model independence $\leftarrow$ low energy universality

- Weak binding: bound state size $\gg$ interaction range

$\rightarrow$ two-channel model with a contact interaction

$\langle q | \hat{V} | \psi_0 \rangle = g_0$ constant

$\rightarrow$ system can be completely specified by $a$ and $r_e$


$\rightarrow$ full (exact) amplitude: only two observable

$f(p) = \left( -\frac{1}{a} + \frac{r_e}{2} p^2 - ip \right)^{-1}$
Compositeness of hadrons and near-threshold bound state

Interpretation of negative effective range

For $Z > 0$, effective range is always negative.

$$a = \frac{2(1 - Z)}{2 - Z} R + \mathcal{O}(R_{\text{typ}}), \quad r_e = \frac{-Z}{1 - Z} R + \mathcal{O}(R_{\text{typ}}),$$

\[
\begin{cases}
  a \sim R_{\text{typ}} \ll -r_e & \text{(elementary dominance)}, \\
  a \sim R \gg r_e \sim R_{\text{typ}} & \text{(composite dominance)}. 
\end{cases}
\]

Simple (e.g. square-well) attractive potential: $r_e > 0$

- only “composite dominance” is possible.

$r_e < 0$: energy- (momentum-)dependence of the potential


- pole term/Feshbach projection of coupled-channel effect

Negative $r_e \rightarrow$ something other than $|p>$: CDD pole
Compositeness theorem

Exact $B \to 0$ limit:

If the s-wave scattering amplitude has a pole exactly at the threshold with a finite range interaction, then the field renormalization constant vanishes.

$$Z(0) \text{ vanishes for } g_0 \neq 0. \text{ If } g_0 = 0, \text{ no pole in the amplitude.}$$

For bare state-continuum model ($c$: nonzero constant)

$$Z(B) = \frac{1}{1 - \Sigma'(-B)} \approx \frac{1}{1 - c \frac{g_0^2}{\sqrt{B}}}$$

$$\text{Im } \Sigma(p^2/2\mu) \propto p^{2l+1}$$

For general potentials: poles in the effective range expansion

$$p_1 = i\sqrt{2\mu B}, \quad p_2 = -i\sqrt{2\mu B} \frac{2 - Z(B)}{Z(B)}$$

If $Z(0) \neq 0$, then both $p_1$ and $p_2$ go to zero for $B \to 0$ : contradict with simple pole at $p=0 \to Z(0)=0$

Compositeness of hadrons and near-threshold bound state

Interpretation of the compositeness theorem

\[ Z(B) : \text{overlap of the bound state with bare state} \]

\[ \left| \langle \Psi | \begin{pmatrix} \psi_0 \\ 0 \end{pmatrix} \rangle \right|^2 + \int \left| \langle \Psi | \begin{pmatrix} 0 \\ q \end{pmatrix} \rangle \right|^2 \, d^3q = 1 \]

- \( Z(B \neq 0) = 0 \) \( \rightarrow \) Bound state is completely composite.

Two-body wave function at \( E = 0 \):

\[ u_{l,E=0}(r) \xrightarrow{r \to \infty} r^{-l} \]

\[ Z(0) = 0 : \text{Bound state is completely composite.} \]

Composite component is infinitely large so that the fraction of any finite admixture of bare state is zero.
Near-threshold resonances

Generalization to resonances

Compositeness of bound states

\[ Z(B) = \frac{1}{1 - \Sigma'(-B)} \]

Naive generalization to resonances:


\[ Z(E_R) = \frac{1}{1 - \Sigma'(-E_R)} \]

- Problem of interpretation (probability?)

<— Normalization of resonances

\[ \langle R | R \rangle \rightarrow \infty, \quad \langle \tilde{R} | R \rangle = 1 \]

\[ 1 = \langle \tilde{R} | B_0 \rangle \langle B_0 | R \rangle + \int dp \langle \tilde{R} | p \rangle \langle p | R \rangle \]

\[ \langle \tilde{R} | B_0 \rangle = \langle B_0 | R \rangle \neq \langle B_0 | R \rangle^* \]

Near-threshold resonances

Weak binding limit for bound states

- Model-independent (no potential, wavefunction, ... )
- Related to experimental observables

What about near-threshold resonances (~ small binding)?

shallow bound state: model-independent structure

general bound state: model-dependent real

general resonance: model-dependent complex
Near-threshold resonances

Poles in the effective range expansion

Near-threshold pole: effective range expansion


\[ f(p) = \left( -\frac{1}{a} + \frac{r_e}{2} p^2 - i p \right)^{-1} \]

\[ p^\pm = \frac{i}{r_e} \pm \frac{1}{r_e} \sqrt{\frac{2r_e}{a} - 1} \]

- pole trajectories with a fixed \( r_e < 0 \)

Resonance pole position \( \leftrightarrow (a, r_e) \)
Near-threshold resonances

**Application: \( \Lambda_c(2595) \)**

**Pole position of \( \Lambda_c(2595) \) in \( \pi \Sigma_c \) scattering**

- central values in PDG

\[
E = 0.67 \text{ MeV}, \quad \Gamma = 2.59 \text{ MeV} \quad p^\pm = \sqrt{2 \mu (E \mp i\Gamma/2)}
\]

- deduced threshold parameters of \( \pi \Sigma_c \) scattering

\[
a = - \frac{p^+ + p^-}{ip^+ p^-} = -10.5 \text{ fm}, \quad r_e = \frac{2i}{p^+ + p^-} = -19.5 \text{ fm}
\]

- field renormalization constant: complex

\[
Z = 1 - 0.608i
\]

Large negative effective range

\(<— \text{ substantial elementary contribution other than } \pi \Sigma_c \text{ (three-quark, other meson-baryon channel, or ... )}\)

\( \Lambda_c(2595) \) is not likely a \( \pi \Sigma_c \) molecule
Near-threshold mass scaling

Hadron mass scaling and threshold effect

Systematic expansion of hadron masses

- ChPT: light quark mass $m_q$
- HQET: heavy quark mass $m_Q$
- large $N_c$: number of colors $N_c$

What happens at two-body threshold?

Hadron mass scaling

$$m_H(x); \quad x = \frac{m_q}{\Lambda}, \frac{\Lambda}{m_Q}, \frac{1}{N_c}$$
Near-threshold mass scaling

**Formulation**

**Coupled-channel Hamiltonian (bare state + continuum)**

\[
\begin{pmatrix}
M_0 & \hat{V} \\
\hat{V} & \frac{p^2}{2\mu}(+\hat{V}_{sc})
\end{pmatrix}
\begin{pmatrix}
\Psi \\
\Psi
\end{pmatrix}
= E \begin{pmatrix}
\Psi \\
\Psi
\end{pmatrix},
\begin{pmatrix}
\Psi \\
\Psi
\end{pmatrix}
= \begin{pmatrix}
\psi_0 \\
\chi_E(p)
\end{pmatrix}
\]

**Equivalent single-channel scattering formulation**

\[
\hat{V}_{eff}(E) = \frac{\hat{V}\psi_0\psi_0\hat{V}}{E - M_0} \sim \ldots
\]

\[
f(p, p', E) = -\frac{4\pi^2\mu \langle p | \hat{V} | \psi_0 \rangle \langle \psi_0 | \hat{V} | p' \rangle}{E - M_0 - \Sigma(E)} \sim \ldots
\]

**Pole condition:**

\[
E_h - M_0 = \Sigma(E_h)
\]

**Question:** How \(E_h\) behaves against \(M_0\) around \(E_h = 0\)?
Near-threshold mass scaling

Bound state condition around $E_h=0$

$$E_h + \Sigma(0) - \delta M = \Sigma(E_h)$$

Leading contribution of the expansion:

$$E_h = \frac{1}{1 - \Sigma'(0)} \delta M = Z(0) \delta M, \quad \Sigma'(E) \equiv \frac{d\Sigma(E)}{dE}$$

Field renormalization constant

$$\left| \langle \Psi | \begin{pmatrix} \psi_0 \\ 0 \end{pmatrix} \rangle \right|^2 + \int \left| \langle \Psi | \begin{pmatrix} 0 \\ q \end{pmatrix} \rangle \right|^2 d^3q = 1$$

$Z(0)$ vanishes for $l=0$: compositeness theorem

$$E_h \propto \begin{cases} \mathcal{O}(\delta M^2) & l = 0 \\ \delta M & l \neq 0 \end{cases}$$
Near-threshold mass scaling

Near-threshold bound state (general)

General argument by **Jost function** (Fredholm determinant)


$$f_l(p) = \frac{\mathcal{J}_l(-p) - \mathcal{J}_l(p)}{2ip\mathcal{J}_l(p)}$$

pole (eigenstate) = Jost function zero

Expansion of the Jost function:

$$\mathcal{J}_l(p) = \begin{cases} 1 + \alpha_0 + i\gamma_0 p + \mathcal{O}(p^2) & l = 0 \\ 1 + \alpha_l + \beta_l p^2 + \mathcal{O}(p^3) & l \neq 0 \end{cases}$$

- $\gamma_0$ and $\beta_l$ are nonzero for a general potential
- **zero** at $p=0$ ($1+\alpha_l=0$) must be **simple** (double) for $l=0$ ($l\neq0$)


Near-threshold scaling:

$$1 + \alpha_l \sim \delta M \quad \Rightarrow \quad E_h \propto \begin{cases} -\delta M^2 & l = 0 \\ \delta M & l \neq 0 \end{cases} \quad (\delta M < 0)$$
Near-threshold mass scaling

General threshold behavior

Near threshold scaling:

- \( \delta M < 0 \)
  \[
  E_h \propto \begin{cases} 
  -\delta M^2 & l = 0 \\
  \delta M & l \neq 0 
  \end{cases}
  \]

- \( \delta M > 0 \)
  \[
  E_h \propto \begin{cases} 
  -\delta M^2 & l = 0 \\
  \text{Re } E_h \propto \delta M \\
  \text{Im } E_h \propto -(\delta M)^{l+1/2} & l \neq 0 
  \end{cases}
  \]

Numerical calculation

\[
\langle q | \hat{V} | \psi_0 \rangle = g_l |q|^l \Theta(\Lambda - |q|)
\]

c.f. NN \(^1\!S_0\)

slope: \( Z(0) \)
Near-threshold mass scaling

Chiral extrapolation across s-wave threshold

s-wave: bound state —> virtual state —> resonance

Near-threshold scaling: nonperturbative phenomenon

—> Naive ChPT does not work; resummation required.

c.f.) NN sector, \( \overline{KN} \) sector, …
Near-threshold mass scaling

Scaling of three-body bound state

Near-threshold scaling is universal for two-body system.
- What about three-body case?


[Diagram showing near-threshold scaling for three-body case with Efimov trimer (s-wave three-body bound state).]
Compositeness / elementariness
- suitable classification for hadron structure
- model independent in the weak binding limit

Near-threshold resonance:
- elementariness from effective range

Near-threshold mass scaling:
- s-wave case is different from the others